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Stability of Ion Acoustic Turbulent States

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Several proposed renormalized theories for "strong" ion acoustic turbulence are compared to the direct interaction approximation. These are applied to the calculation of the stability of ion acoustic turbulent states to the excitation of Langmuir waves. A kinetic instability proposed by Tsytovich, Stenflo, and Wilhelmsson is shown to be stabilized by resonant and nonresonant decay processes. The global kinetic stability of the Langmuir spectrum is enhanced through the coupling of opposite phase velocity Langmuir waves by the decay processes and by nonresonant scattering to regions of strong Landau damping.

KEY WORDS: Ion-acoustic waves; Langmuir waves; turbulence; statistical theory; renormalized theory; plasma instability.

1. INTRODUCTION

Ion acoustic turbulence is probably the best understood regime of collisionless plasma turbulence. Quite detailed experiments are available as discussed in the articles by Slusher[(1)] and Horton,[(2)] and there is a qualitative and perhaps semiquantitative agreement between experiment and theory as discussed in the article by Horton.[(2)] Indeed, largely because of the efforts of Horton and Choi,[(3)] the renormalized turbulence theory for ion acoustic turbulence has been evaluated in more detail than for any other branch of collisionless plasma turbulence. The improvement of numerical simulation techniques discussed by Lindman[(4)] promises to provide a tool for more precise comparisons with analytic statistical turbulence theories.

This article is mainly devoted to a study of the stability of states of ion acoustic turbulence to the excitation of high-frequency Langmuir waves.

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This is essentially a side issue for ion acoustic turbulence unless—as predicted by some workers[5]—Langmuir waves could be unstable with appreciable growth rates, implying that the saturation of Langmuir excitations should be considered in formulating a theory of ion acoustic turbulence. The efficient transfer of energy from low-frequency ion acoustic excitations to high-frequency Langmuir waves would have implications in many areas and has particularly excited the interest of some astrophysicists since many mechanisms exist for the creation of ion acoustic turbulence and it is appealing to have a mechanism to convert this energy into high-frequency radiation.

The stability of a state of ion acoustic turbulence to Langmuir wave excitation, not surprisingly, depends on the details of the ion acoustic turbulence. Among these are the shape of the ion acoustic potential fluctuation spectrum in wave-vector space, the stage of evolution of the turbulence, and the source of free energy which drives the turbulence. This brings in the study of the self-consistent dielectric response function for ion acoustic turbulence, which is intrinsic to the study of ion acoustic turbulence and has been discussed in some detail by Horton.

Since the work of Kraichnan[6] in 1958 it is known that there are two self-consistent aspects of a statistical turbulence theory. First one needs the linear (or infinitesimal) response function $R_{\mathbf{k}}(t, t')$ (or Green's function) which measures the mean response at time t of the turbulent system to an infinitesimal mean perturbation with wave vector $\mathbf{k}$ at time t'. In quasistationary turbulence the fourier transform over $t - t'$ of this function, $R_k(\omega)$, has poles in the complex frequency plane at $\omega = \omega_k + i\gamma_k$ which describe the coherent excitations of the turbulent system. The sign of γ_k determines the stability of these excitations. The calculation of this turbulent response function is self-consistently tied to the calculation of the two-point correlation function of the fluctuating fields, $C_{\mathbf{k}}(t, t') = \langle \delta f_{\mathbf{k}}(\mathbf{v}, t)\, \delta f_{\mathbf{k}}(\mathbf{v}', t') \rangle$, where for Vlasov turbulence $\delta f_{\mathbf{k}}(\mathbf{v}, t)$ is the fluctuation from the mean of the phase space distribution function. Here angular brackets denote ensemble averages.

If the prototype nonlinear equation (e.g., the Vlasov equation) has the form

$$(\partial_t + L_{k_1})\, \delta f_{k_1} = \frac{1}{2} \sum_{k_1 = k_2 + k_3} a_{k_1 k_2 k_3}\, \delta f_{k_2}\, \delta f_{k_3} \tag{1.1}$$

where $a_{k_1 k_2 k_3} = a_{k_1 k_3 k_2}$, representing a quadratically nonlinear equation, Kraichnan[7] and Leith[8] have shown that the fluctuations obey a generalized Langevin equation:

$$(\partial_t + L_k)\, \delta f_k(t) + \int_0^t dt'\, \nu_k^{\mathrm{d}}(t, t')\, \delta f_k(t') = s_k^{\mathrm{nd}}(t) \tag{1.2}$$

where v_k^{d} is a renormalization of the linear operator L_k resulting from a consolidation of diagonal terms proportional to δf_k and the right-hand side is an effective Gaussian noise source arising from terms not coherent with δf_k—i.e., not proportional to δf_k. The direct interaction approximation (DIA) arises from the lowest order of the algorithm determining v_k^{d} and s^{nd}:

$$v_{k_1}^{\mathrm{d}}(t, t') = \sum_{k_1 = k_2 + k_3} a_{k_1 k_2 k_3} a_{k_2 k_3 k_1} R_{k_2}(t, t')\, C_{k_3}(t, t') \tag{1.3}$$

and

$$\langle s_{k_1}^{\mathrm{nd}}(t)\, s_{k_1}^{\mathrm{nd}}(t') \rangle = \tfrac{1}{2} \sum_{k_1 = k_2 + k_3} |a_{k_1 k_2 k_3}|^2\, C_{k_2}(t, t')\, C_{k_3}(t, t') \tag{1.4}$$

The response function $R_k(t, t')$ is the causal Green's function of the diagonal part of the Langevin operator

$$(\partial_t + L_k)\, R_k(t - t') + \int_0^t dt''\, v_k^{\mathrm{d}}(t, t'')\, R_k(t'' - t) = \delta(t - t') \tag{1.5}$$

By inversion of Eq. (1.2) using this operator one finds in Fourier space for the time difference variable $\{R_k(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} d(t - t') \exp[i\omega(t - t')]\, R_k(t - t'), \text{etc.}\}$

$$\langle |\delta f|_{k\omega}^2 \rangle = |R_k(\omega)|^2 \langle |s^{\mathrm{nd}}|_{k\omega}^2 \rangle \tag{1.6}$$

which is manifestly positive definite.

The response function is also seen to be the coherent response of the system to an infinitesimal mean source. Suppose a source $s_k^{\mathrm{ext}}(t)$ whose ensemble average $\langle s^{\mathrm{ext}}(t) \rangle$ is nonzero is added to the right-hand side of Eq. (1.2); then it is easy to see that R_k can be expressed as the functional derivative

$$R_k(t - t') = \left. \frac{\delta \langle \delta f_k(t) \rangle}{\delta \langle s_k^{\mathrm{ext}}(t') \rangle} \right|_{s_k^{\mathrm{ext}} = 0}$$

The fluctuations are driven by the incoherent noise source in Eq. (1.2) and respond with a turbulence renormalized response represented by $R_k(t - t')$. The incoherent noise source is related to the intrinsic stochasticity of the turbulent system. The renormalized response as determined by the poles of $R_k(\omega)$ can often (but not always) be expressed in terms of renormalized frequencies and damping rates of the linear modes of the system.

In this paper we will apply these ideas to several aspects of ion acoustic turbulence. In Section 2 we review several proposed renormalized theories of "strong" plasma turbulence and consider their relationship to the DIA approach. Only the "simply renormalized" theory of Horton and Choi[3] has been worked out in detail for ion acoustic turbulence. This type of theory is based on the underlying assumption that the electrostatic fluctuations are (quasi) Gaussian random variables. This assumption is dropped in the full DIA theory but leads to a possibly intractable set of equations which account for the self-consistent connection between electrostatic fluctuations and fluctations in the charged particle phase-space distributions; because of the nonlinearity of the Vlasov equation neither of these types of fluctuations can be quasi-Gaussian.

In Section 3 we apply the renormalized turbulence equations to the stability of Langmuir waves in states of ion acoustic turbulence. The question of stability centers around a kinetic instability mechanism arising in weak turbulence theory which was first identified by Tsytovich, Stenflo, and Willhelmsson[5] (TSW). They called this effect "turbulent bremsstrahlung"; in this mechanism electrons which are in resonance with the low-frequency ion acoustic waves are retarded and, owing to their accelerated motion, radiate a variety of high-frequency waves including electromagnetic and Langmuir waves.

We have studied the competition of the "turbulent bremsstrahlung" (TB) effect with other effects which tend to stabilize Langmuir (and electromagnetic) waves. We find that decay interactions, in which a Langmuir wave is scattered into another wave vector state by the emission or absorption of an ion acoustic wave, generally stabilize Langmuir waves against TB. In the results to be discussed here we extend our earlier work[9] to include the strong turbulence effects in stationary turbulence and the global wave kinetics of the decay process which takes into account scattering-out as well as scattering-in processes. Here a new stabilizing effect arises: The TB effect destabilized Langmuir waves with (say) positive phase velocities while stabilizing waves with negative phase velocities. The decay process strongly couples waves with the two signs of the phase velocity, and in this coupled system the net destabilizing effect is greatly reduced. We have shown that nonresonant decay processes provide stabilization in regimes where resonant processes are not kinematically allowed and provide a mechanism by which Langmuir waves are scattered to high wave numbers where Landau damping provides the ultimate stabilization of the Langmuir spectrum. The consideration of nonresonant decay processes implies a parameter ordering which accounts for the finite width of spectral resonances and retains terms not customarily considered in weak turbulence theory; these terms generally dominate the TB effect.

2. PLASMA TURBULENCE THEORY: WEAK AND RENORMALIZED

We begin with the iterative solution of the Vlasov–Poisson system as given by Horton's[2] Eq. (7) to third-order terms:

$$\varepsilon^{1}_{k}\phi_{k} + \sum_{k_1+k_2=k} \varepsilon^{(2)}_{k_1,k_2}\phi_{k_1}\phi_{k_2} + \sum_{k_1+k_2+k_3=k} \varepsilon^{(3)}_{k_1,k_2,k_3}\phi_{k_1}\phi_{k_2}\phi_{k_3} = 0 \quad (2.1)$$

where $k_i = (\mathbf{k}_i, \omega_i)$, etc. The DIA algorithm, suitably generalized to account for the cubic nonlinearity, yields the following equation for the electrostatic fluctuation spectrum $I_k = \langle |\phi_k|^2 \rangle$:

$$|\varepsilon^{\mathrm{nl}}_{k}|^2 \langle |\phi_k|^2 \rangle = 2 \sum_{k=k_1+k_2} |\varepsilon^{(2)}_{k_1,k_2}|^2 \langle |\phi|^2_{k_1} \rangle \langle |\phi|^2_{k_2} \rangle \quad (2.2)$$

where the renormalized dielectric response is

$$\varepsilon^{\mathrm{nl}}_{k} = \varepsilon^{1}_{k} - \sum \left[\frac{4\varepsilon^{(2)}_{k',k-k}\varepsilon^{(2)}_{-k',k}}{\varepsilon^{\mathrm{nl}}_{k-k'}} - 2\varepsilon^{(3)}_{k',-k',k} \right] \langle |\phi|_{k'} \rangle^2 \quad (2.3)$$

This is Horton's Eq. (22), which has first derived by Tsytovich[10] for plasma turbulence. The nonlinear coupling coefficients will be given explicitly below.

These equations will be the basis for most of our discussions in this paper. Several observations are important to make: First we note as did Horton[2] that the simple weak turbulence theory results in Eq. (2.3) but with $\varepsilon^{\mathrm{nl}}_{k-k'}$ replaced by the linear dielectric $\varepsilon^{1}_{k-k'}$ and Eq. (2.2) with $|\varepsilon^{\mathrm{nl}}_{k}|^2$ replaced by $\varepsilon^{\mathrm{nl}}_{k}(\varepsilon^{1}_{k})^*$. Various arguments including the summation of secular terms result in the more symmetrical expressions given which are clearly consistent with the general realizability structure of the DIA. The second observation is that the incoherent source term, arising in this approximation from wave–wave coupling, is present as in the general DIA. In the strict weak turbulence theory such as that of Kadomtsev[11] this term is taken to be zero because there is no resonant decay interaction between ion acoustic waves. This leads to the condition $\varepsilon^{\mathrm{nl}}_{k}\{\langle |\phi|^2_{k} \rangle\} = 0$, which determines the spectrum in steady state or for the quasistationary case to Horton's Eq. (9) with the final term on the right-hand side omitted. The mode simulation studies of Horton *et al.*[3] indicate that these three wave incoherent source terms are subdominant for determining the ion acoustic turbulent spectrum.

Another observation to be made is that while Eq. (2.2) is exactly the result of the DIA algorithm applied to Eq. (2.1) (suitably generalized to incorporate the cubic nonlinearity) it is *not* the same as the DIA algorithm applied to the full quadratically nonlinear Vlasov equation! The later theory treats $\delta f^{j}_{k}(\mathbf{v}, t)$ and $\delta\phi_k(t) = \sum_j k^{-2} 4\pi e_j \int d\mathbf{v}\, \delta f^{j}_{k}(\mathbf{v}, t)$ as statistically equivalent; this is consistent with this linear relationship between δf^j and $\delta\phi$. (The index j represents the charge species and will sometimes be suppressed.) The nonlinear Poisson equation is derived from an expansion of δf_k in powers of $\delta\phi_k$, which implies that δf_k and $\delta\phi_k$ have different statistics. For a trivial example, if we assume $\delta\phi_k$ to obey Gaussian statistics then δf_k, if δf_k is determined as a power series in ϕ_k, cannot be Gaussian beyond the linear approximation.

The full DIA applied to the Vlasov equation has been given in several references.[12–17] It is a very complicated theory to evaluate even for the simplest cases; in the present article we will only touch on certain aspects which can be seen by direct inference from Eqs. (2.2) and (2.3); this will give some of the flavor of the complete theory but will be rather incomplete.

For this and other reasons we need the complete expressions for the nonlinear coupling coefficients which can be found in Ref. 3 and in most standard texts on weak turbulence theory:

$$\begin{aligned}\varepsilon^{(2)}_{k',k-k'} = {} & -\frac{1}{2}\sum_j \frac{e_j}{m_j}\frac{\omega_{pj}^2}{k^2} \\ & \times \int d\mathbf{v}\,\frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\delta}\left(\mathbf{k}'\cdot\partial_v \frac{1}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i\delta}\,\mathbf{k}''\cdot\partial_v \right. \\ & \left. + \mathbf{k}''\cdot\partial_v \frac{1}{\omega' - \mathbf{k}'\cdot\mathbf{v} + i\delta}\,\mathbf{k}'\cdot\partial_v\right)\langle f^j(\mathbf{v})\rangle \end{aligned} \tag{2.4}$$

$$\begin{aligned}\varepsilon^{(2)}_{k,-k'} = {} & -\frac{1}{2}\sum_j \frac{e_j}{m_j}\frac{\omega_{pj}^2}{k^2} \\ & \times \int d\mathbf{v}\,\frac{1}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i\delta}\left(-\mathbf{k}'\cdot\partial_v \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\delta}\,\mathbf{k}\cdot\partial_v \right. \\ & \left. + \mathbf{k}\cdot\partial_v \frac{1}{\omega' - \mathbf{k}'\cdot\mathbf{v} - i\delta}\,\mathbf{k}'\cdot\partial_v\right)\langle f^j(\mathbf{v})\rangle \end{aligned} \tag{2.5}$$

$$\begin{aligned}\varepsilon^{(3)}_{k',-k',k} = {} & -\frac{1}{2}\sum_j \left(\frac{e_j}{m_j}\right)^2 \frac{\omega_{pj}^2}{k^2} \\ & \times \int d\mathbf{v}\,\frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\delta}\,\mathbf{k}'\cdot\partial_v \frac{1}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i\delta} \\ & \times \left(-\mathbf{k}\cdot\partial_v \frac{1}{\omega' - k'\cdot v - i\delta}\,\mathbf{k}'\cdot\partial_v \right. \\ & \left. + \mathbf{k}'\cdot\partial_v \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\delta}\,\mathbf{k}\cdot\partial_v\right)\langle f^i(\mathbf{v})\rangle \end{aligned} \tag{2.6}$$

On very general grounds it has been proven that the nonlinear dielectric response can always be written in the form[14,12]

$$\tilde{\varepsilon}_k^{\rm nl} = 1 - \sum_{j=e,i} i\omega_{p_j}^2 k^{-2} \int d\mathbf{v} \int d\mathbf{v}'\, g_k^j(\mathbf{v}, \mathbf{v}')\mathbf{k} \cdot \partial_{v'} \bar{f}_k^j(\mathbf{v}') \tag{2.7}$$

where $g^j_{\mathbf{k},\omega}(\mathbf{v}, \mathbf{v}')$ is a renormalized single-particle (or quasiparticle) propagator for particle of species j; $g^j_{\mathbf{k},\omega}(\mathbf{v}, \mathbf{v}')$ is generally nonlocal in velocity space. Less familiar is the "renormalized distribution," which obeys an equation

$$\mathbf{k} \cdot \partial_v \bar{f}^j_{\mathbf{k},\omega}(v) = \mathbf{k} \cdot \partial_v \langle f^j(\mathbf{v}) \rangle + \mathbf{k} \cdot \mathbf{v}^{E,j}_{\mathbf{k},\omega}(\mathbf{v}) \tag{2.8}$$

This nomenclature is used because Eq. (2.7) is reminiscent of the linear dielectric response where $g_{\mathbf{k}\omega}(\mathbf{v}, \mathbf{v}') = i(\omega - \mathbf{k} \cdot \mathbf{v} + i\delta)^{-1}\, \delta(\mathbf{v} - \mathbf{v}')$ and $\bar{f}^j_{k\omega} = \langle f^j(\mathbf{v}) \rangle$.

This is a very convenient notation but is quite misleading physically: $\bar{f}_k(\mathbf{v})$ is not the renormalized or observable mean distribution which is simply $\langle f \rangle$ in our notation: It is easily shown that $v_k^E = 0$ if $\partial_v \langle f \rangle = 0$. The nonzero mean distribution $\partial_v \langle f \rangle$ or "mean field" provides a linear coupling of the fluctuation f_k to the electrostatic field fluctuations ϕ_k (or E_k); the operator $v_k^E(\mathbf{v})$ describes the renormalization of this mean-field-induced coupling or "vertex" in field theory language. This physical effect was apparently first considered by Kadomtsev[11,15] and later treated in detail by DuBois and Espedal[12] for Vlasov turbulence and in a special approximation for drift wave turbulence by Dupree and Tetrault.[18] The full theory of $g_{\mathbf{k}}(\mathbf{v}, \mathbf{v}')$ and $\bar{f}_k(\mathbf{v})$ has been given elsewhere.[12,15] It is shown that $g_k(\mathbf{v}, \mathbf{v}')$ obeys an equation of the form

$$\begin{aligned} -ig_k(\mathbf{v}, \mathbf{v}') &= [\omega - \mathbf{k} \cdot \mathbf{v} + iv^f(\mathbf{v}, \mathbf{v}')]^{-1}\, \delta(\mathbf{v} - \mathbf{v}') \\ &= \frac{\delta(\mathbf{v} - \mathbf{v}')}{\omega - \mathbf{k} \cdot \mathbf{v} + i\delta} - \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v} + i\delta}\, iv_k^f(\mathbf{v}, \mathbf{v}') \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}' + i\delta} \\ &\quad - \int d\mathbf{v}'' \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v} + i\delta}\, \mathbf{v}_k^f(\mathbf{v}, \mathbf{v}'') \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}'' + i\delta} \\ &\quad \times v_k^f(\mathbf{v}'', \mathbf{v}') \frac{1}{\omega - \mathbf{k} \cdot \mathbf{v}' + i\delta} + \cdots \end{aligned} \tag{2.9}$$

where $v_k^f(\mathbf{v}, \mathbf{v}')$ is a single-particle "self-energy" operator which accounts for the reaction of the turbulent system on the propagation of a single-particle excitation. This is a generalization of Eq. (13) in Horton's article. If we use this expansion of $g_k(\mathbf{v}, \mathbf{v}')$ we can write $\tilde{\varepsilon}_k^{\rm nl}$ as

$$\tilde{\varepsilon}_k^{\text{nl}} = 1 + \sum_j \omega_{p_j}^2 k^{-2} \int d\mathbf{v} \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\delta} \mathbf{k}\cdot\partial_v \langle f^j(\mathbf{v})\rangle$$
$$- i \sum_j \omega_{p_j}^2 k^{-2} \int d\mathbf{v} \int d\mathbf{v}' \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\delta} v_f(\mathbf{v}, \mathbf{v}'')$$
$$\times \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v}' + i\delta} \mathbf{k}\cdot\partial_v \langle f^j(\mathbf{v})\rangle$$
$$+ \sum_j \omega_{p_j}^2 k^{-2} \int d\mathbf{v} \frac{1}{\omega - \mathbf{k}\cdot\mathbf{v} + i\delta} \mathbf{k}\cdot\mathbf{v}_{\mathbf{k}\omega}^{E,j}(\mathbf{v}) + \cdots \tag{2.10}$$

On comparison of this equation and the weak turbulence expressions for Eqs. (2.3)–(2.6) for $\varepsilon_k^{\text{nl}}$ we can identify the lowest-order expressions for operators $v_{\mathbf{k}\omega}^f(\mathbf{v}, \mathbf{v}')$ and $\mathbf{v}_{\mathbf{k}\omega}^E(\mathbf{v})$. For example, extracting all terms from Eq. (2.3) which contain two propagators with labels $\mathbf{k}$, ω, we can identify the operator $v_{k\omega}^f(\mathbf{v}, \mathbf{v}')$ as

$$v_{\mathbf{k}\omega}^{f,j}(\mathbf{v}, \mathbf{v}') = \frac{e_j^2}{m_j^2} \int dk' \, \langle |\phi|_{k'}^2 \rangle \mathbf{k}'\cdot\partial_v \frac{i}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i\delta} \mathbf{k}'\cdot\partial_v \delta(\mathbf{v} - \mathbf{v}')$$
$$+ \frac{e_j^2}{m_j^2} \frac{\omega_{p_j}^2}{k^2} \int dk' \, \frac{\langle |\phi|_{k'}^2 \rangle}{\varepsilon_{k''}^1}$$
$$\times i \left[\mathbf{k}'\cdot\partial_v \frac{1}{\omega'' - \mathbf{k}''\cdot\mathbf{v} + i\delta} \mathbf{k}''\cdot\partial_v \langle f^j(\mathbf{v})\rangle \right.$$
$$\left. + \mathbf{k}''\cdot\partial_v \frac{1}{\omega' - \mathbf{k}'\cdot\mathbf{v} + i\delta} \mathbf{k}'\cdot\partial_v \langle f^j(\mathbf{v})\rangle \right]$$
$$\times \frac{1}{\omega'' - \mathbf{k}''\cdot\mathbf{v}' + i\delta} \mathbf{k}'\cdot\partial_{v'} \tag{2.11}$$

The first term alone leads to the familiar resonance broadened propagator in the simply renormalized approximation given in Horton's Eq. (12). In this approximation v^f is a local second derivative operator in v space. (However, the operator $g_k(\mathbf{v}, \mathbf{v}')$ is nonlocal even in this approximation.) The second term is nonlocal in velocity space and arises from the polarization terms in the weak turbulence theory. The existence of such nonlocal renormalization terms arising from polarization effects was first pointed by DuBois and Espedal.[12] The complete DIA algorithm[15] gives a "completely renormalized" version of Eq. (2.11) containing renormalized propagators and dielectric responses in all expressions as well as more general spectral correlation functions. The consequences of the nonlocal polarization terms are poorly understood for ion ion acoustic turbulence. The simply renormalized scheme without polarization effects used by Horton and Choi[3] has the great advantage of being amenable to detailed evaluation and leads to qualitative agreement with experiment and computer simulation. It remains a challenge to estimate the effect of the polarization terms for ion acoustic turbulence. The only case where analytic progress has been made is for the nonlinear evolution of the weak warm beam instability for an electron plasma in the strong mode coupling regime.[19] In this case it was shown that the polarization effects are exactly the same order as the direct renormalization terms.

In considering the problem of the stability of Langmuir waves in a state of ion acoustic turbulence in Section 3 we find that the *perturbative* polarization corrections to the dielectric response play a crucial role.

The renormalization of the mean field coupling $v_k^E(v)$ can also be extracted from the weak turbulence equation for $\varepsilon_k^{\mathrm{nl}}$ by looking at the terms containing only one factor $(\omega - k \cdot v + i\delta)^{-1}$ and comparing with Eq. (2.10). One finds

$$
\begin{aligned}
\mathbf{k}\cdot\mathbf{v}^{E,j}_{\mathbf{k},\omega}(\mathbf{v}) = {} & \frac{e_j^2}{m_j^2}\int dk'\, \mathbf{k}'\cdot\partial_v \frac{1}{\omega''-\mathbf{k}''\cdot\mathbf{v}+i\delta}\,\mathbf{k}\cdot\partial_v \\
& \times \frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}-i\delta}\,\mathbf{k}'\cdot\partial_v\langle f^j\rangle\langle|\phi_{k'}|^2\rangle \\
& + \frac{e_j^2}{m_j^2}\frac{\omega_{p_j}^2}{k^2}\int dk' \int d\mathbf{v}'\,\frac{\langle|\phi_{k'}|^2\rangle}{\varepsilon_{k''}}\left[\mathbf{k}'\cdot\partial_v\frac{1}{\omega''-\mathbf{k}''\cdot\mathbf{v}+i\delta}\,\mathbf{k}''\cdot\partial_v\right. \\
& \left. + \mathbf{k}''\cdot\partial_v\frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}+i\delta}\,\mathbf{k}'\cdot\partial_v\right]\langle f^j(\mathbf{v})\rangle \\
& \times \frac{1}{\omega''-\mathbf{k}''\cdot\mathbf{v}'+i\delta}\,\mathbf{k}\cdot\partial_{v'}\frac{1}{\omega'-\mathbf{k}'\cdot\mathbf{v}'-i\delta}\,\mathbf{k}'\cdot\partial_{v'}\langle f^j(\mathbf{v}')\rangle
\end{aligned}
$$

Again we see there is a direct and a polarization contribution involving the dielectric response. The direct term was actually first correctly calculated by Kadomtsev[11] and subsequently by several other authors[12,18] and corresponds to the renormalization $\mathscr{E}_k\langle v\rangle$ given in Horton's Eqs. (14) and (15). The polarization terms were first considered in Ref. 12 in the fully renormalized DIA theory. The theory of Kadomtsev also had polarization contributions but they did not correspond exactly to the DIA results; recent work[15] has shown that when his "weak coupling approximation" is consistently carried out it leads exactly to the DIA results. The effects of the mean field coupling renormalization have not been systematically explored for ion acoustic turbulence. Again for the weak beam instability in one dimension these terms were found to be of the same order as the propagator renormalization effects.[19] The renormalized theory for g and v^E without the polarization terms can be shown[17] to arise from the full Vlasov equation (not the expansion in powers of ϕ) when the electric potential fluctuations are *assumed* to be Gaussianly correlated.

We can then identify three turbulent theories:

Theory 1. The DIA algorithm applied to the nonlinear Poisson equation (2.1) which leads to the Tsytovich equation (2.2) with renormalized $\varepsilon^{\rm nl}_{k-k'}$ but no renormalized g's or v^E's in its internal structure. This theory we have seen contains the leading terms in the renormalization series for g and v^E to order $\langle|\phi|^2\rangle$. We will use this theory extensively in Section 3.

Theory 2. The theory of the Vlasov equation which *assumes Gaussian correlated* $\phi_{\mathbf{k}}$ and which leads to the renormalized g and v^E without polarization terms.[17] This is the theory evaluated in the simply renormalized case but with $v^E = 0$ by Horton *et al.*[3] for ion acoustic turbulence.

Theory 3. The full DIA for the Vlasov–Poisson system which treats the non-Gaussian correlations of δf and $\delta\phi$ on an equal footing which is consistent with the linear Poisson equation connecting $\delta\phi$ with δf.

Theory 1 has been used by Tsytovich to treat the turbulence-induced finite linewidth of ion acoustic excitations in the quasistationary turbulent state. The finite renormalized linewidth ($\Delta\omega_k$ in Horton, $\tilde{v}^s_k$ in our notation) enters via the renormalized dielectrics $\varepsilon^{\rm nl}_{\mathbf{k}\omega}$ and $\varepsilon^{\rm nl}_{\mathbf{k}-\mathbf{k}',\omega-\omega'}$ in Eqs. (2.2) and (2.3) and in the Lorentzian line shape for $\langle|\phi_{\mathbf{k}\omega}|^2\rangle$ given in Horton's Eq. (23) [or our Eq. (3.9) below]. These turbulence-induced finite linewidths allow off-resonance three-ion acoustic wave interactions to contribute both to the renormalized ion acoustic growth rate $\tilde{v}^s_{\mathbf{k}} \equiv (\partial\varepsilon^{\rm nl}/\partial\omega^s_{\mathbf{k}})^{-1}$ Im $\varepsilon^{\rm nl}(\mathbf{k}, \omega^s_{\mathbf{k}})$ and to the incoherent noise on the right-hand side of Eq. (2.2). Tsytovich finds $\tilde{v}^s/\omega^s \sim (W^s/nT)$ when $(k'\lambda_D)^4_{\rm max} > W^s/nT > (k'\lambda_D)^4_{\rm min}$; in a strong resonance broadened limit we find $\tilde{v}^s/\omega^s \sim (W^s/nT)^{1/2}$ when $(k'\lambda_D)^4_{\rm max} < W^s/nT$. Here $k'_{\rm min}$ and $k'_{\rm max}$ are, respectively, the minimum and maximum wave numbers in the support of the ion acoustic spectrum. (A similar calculation for Langmuir waves is carried out in Section 3.1.) In our notation T represents the electron temperature and λ_D the electron Debye length.

In the quasi-Gaussian Theory 2 the incoherent noise contribution is changed from that in Eq. (2.2) when wave–particle resonances become important in the wave–wave matrix elements such as $\varepsilon^{(2)}_{k_1k_2}$ and higher-order terms. This leads to the equations for the two-point phase space correlation function considered by Dupree in his "clump" theory.[17] Consideration of this subject in detail is beyond the scope of this paper. Enhanced incoherent noise due to the chaotic granulation of phase space will tend to *increase* the value of the renormalized ion acoustic damping $|\tilde{\nu}^s_{k'}|$ in the quasistationary case over that predicted by Theory 1. We consider only well-developed ion acoustic turbulence dominated by ion acoustic wave fluctuations.

Theory 3, the full Vlasov DIA,[15] is so complex that it has not been applied in detail to the ion acoustic turbulence problem. Polarization terms complicate both the response ε^{nl} and the incoherent source term introducing significant differences from Theories 1 and 2. Some aspects of this theory will be discussed below in the context of the stability of Langmuir waves. The theoretical foundations of the complete Vlasov DIA make it attractive and a challenge for further study.

3. STABILITY OF ION ACOUSTIC TURBULENT STATES TO THE EXCITATION OF LANGMUIR WAVES

In the Introduction we mentioned the current interest in plasma astrophysics in the stability of states of ion acoustic turbulence to the excitation of Langmuir waves. It is not surprising that this stability depends on certain detailed properties of the ion acoustic turbulence, even though this fact has often not been carefully recognized in the literature. The main conclusion of our work is that Langmuir waves are stabilized under a wide range of conditions by resonant or nonresonant decay interactions which scatter Langmuir waves into another wave-vector state by interaction with an acoustic wave.

The theory of the coherent dielectric response of Langmuir waves is formulated in Section 3.1. The destabilizing contribution first proposed by Tsytovich, Stenflo, and Willhelmsonn[5] (TSW) is derived and shown to result from the anisotropy in the turbulent state between positive and negative phase velocity ion acoustic waves.

The decay processes arising from the polarization terms in the coherent dielectric response are considered next. For sufficiently weak levels of ion acoustic turbulence, as for example in the early developing stage, the weak turbulence theory can be used for this calculation, and it is found that the coherent response of Langmuir excitations is generally stable because of relatively strong resonant or nonresonant decay interactions ($L \to L' + s'$). For the Horton–Choi spectrum[3] generated by a drifted distribution (constant current) it is easily seen that resonant decay processes dominate and stabilize the system against the possibly destabilizing effect of TSW. For this spectrum the resonant decay is not allowed for Langmuir waves of sufficiently low k but it is found here that nonresonant decay processes are still sufficient to stabilize the system. These nonresonant processes in the weak turbulence regime depends on the damping rate of ion acoustic excitations in the stationary state (or their growth rate in the case of evolving turbulence). Strong turbulence effects considered within the DIA algorithm are considered here for the first time and shown to increase the stability of the coherent response of Langmuir waves.

The global stability of Langmuir waves is considered in Section 3.2 in a wave-kinetic theory which accounts for the potentially destabilizing scattering-in effects of the decay interactions. Resonant decay interactions strongly couple Langmuir waves with phase velocities destabilized by the TSW effect to opposite signed phase velocities, which are additionally stabilized by this effect. Therefore the net effect of the wave kinetics is a strongly reduced growth rate due to the TSW effect, which can easily be stabilized by relatively weak nonresonant processes that can scatter Langmuir waves into regimes of shorter wavelength where they can be stabilized by Landau damping.

We will emphasize the stability analysis of the ion acoustic turbulence states generated by a constant current, drifted Maxwellian distribution. This is the case treated in detail by Horton and Choi.[3] The spectrum $I^s_{\mathbf{k}'}$ of electrostatic fluctuations is peaked for $\mathbf{k}'$ vectors in the direction of the drift velocity $\mathbf{u}$ and is cylindrically symmetric about this directions. Spectral shapes for this case are shown in the articles by Horton[1] and Slusher.[2] Simulations[1,3,4] show that the temporal development of the turbulence may be pulselike with the level of turbulence growing to a maximum and then decaying because of fast ion acceleration and at long times reaching a lower quasistationary level.

3.1. Coherent Response

The stability of the ion acoustic turbulence system to the coherent excitation of Langmuir waves by an infinitesimal coherent source at the Langmuir frequency can be studied by investigating the roots in the complex ω plane of the renormalized dielectric response, $\varepsilon^{\rm nl}(\mathbf{k}, \omega)$, which was discussed in Section 2. We base our calculations on the direct interaction algorithm, which leads to Eq. (2.3) and which reduces to ordinary weak turbulence theory when $\varepsilon^{\rm nl}(k-k')$ is replaced by the linear dielectric. This renormalization of $\varepsilon^{\rm nl}(k-k')$ plays an important role in the stability calculation for higher values of the ion sound turbulent energy W^s; the inclusion of these effects is necessary to extend the results of Ref. 9 to higher values of W^s.

The turbulence renormalized Langmuir frequency $\omega^{\rm nl}_{\mathbf{k}}$ and damping $\gamma^{\rm nl}_{\mathbf{k}}$

can be found from the zeros of the dielectric response: $\varepsilon^{\rm nl}(\mathbf{k}, \omega_{\mathbf{k}}^{\rm nl} - i\gamma_{\mathbf{k}}^{\rm nl}) = 0$. If we assume $|\gamma_{\mathbf{k}}^{\rm nl}| \ll |\omega_{\mathbf{k}}^{\rm nl}|$ we find in the usual way $\operatorname{Re} \varepsilon^{\rm nl}(\mathbf{k}, \omega_{\mathbf{k}}^{\rm nl}) = 0$ and

$$\gamma_{\mathbf{k}}^{\rm nl} = \frac{\partial \varepsilon^{\rm nl}}{\partial \omega} (\mathbf{k}, \omega_{\mathbf{k}}^{\rm nl})^{-1} \operatorname{Im} \varepsilon^{\rm nl}(\mathbf{k}, \omega_{\mathbf{k}}^{\rm nl}) \tag{3.1}$$

In using this formula we will make the approximation

$$\partial \varepsilon^{\rm nl}/\partial \omega \simeq \partial \varepsilon^{\rm l}/\partial \omega \simeq 2\omega_p^2/\omega^3 \simeq 2/\omega_p \tag{3.2}$$

Near the Langmuir zeros we are then assuming that $\varepsilon^{\rm nl}(\mathbf{k}, \omega)$ has the form

$$\varepsilon^{\rm nl}(\mathbf{k}, \omega) = \frac{2}{\omega_p} (\omega - \omega_{\mathbf{k}}^{\rm nl} + i\gamma_{\mathbf{k}}^{\rm nl}) \tag{3.3}$$

In our work we will explicitly neglect the renormalization of the Langmuir frequency and set $\omega_{\mathbf{k}}^{\rm nl} \simeq \omega_{\mathbf{k}}^{\rm l}$. This will be checked for consistency at various stages. As in the linear problem the perturbed Langmuir frequencies will split into a positive phase velocity branch $\omega_{\mathbf{k}}^{\rm nl} = \omega_{\mathbf{k}}^{\rm l+}$, where $\omega_{\mathbf{k}}^{\rm l+}/k > 0$ and a negative phase velocity branch $\omega_{\mathbf{k}}^{\rm nl} = \omega_{\mathbf{k}}^{\rm l-}$, where $\omega_{\mathbf{k}}^{\rm l-}/k < 0$. When $W^s/nT \ll 1$ we can approximate $\omega_{\mathbf{k}}^{\rm l\pm}$ by the linear values

$$\omega_{\mathbf{k}}^{\rm l\pm} = \pm(\omega_p^2 + 3k^2 v_e^2)^{1/2} \operatorname{sgn} \mathbf{k}, \qquad \operatorname{sgn} \mathbf{k} \equiv \mathbf{k} \cdot \mathbf{u}/|\mathbf{k} \cdot \mathbf{u}| \tag{3.4}$$

where $\mathbf{u}$ is a vector in some convenient direction which we take to be the axis of symmetry of the ion acoustic turbulence (e.g., the drift direction); this then explicitly imposes the usual convention $\omega^{\rm l\pm}_{-\mathbf{k}} = -\omega_{\mathbf{k}}^{\rm l\pm}$ for each phase velocity branch. We then find that $\gamma^{\rm nl}$ can be written formally as the sum of three terms corresponding to the three terms in Eq. (2.3). For the positive phase velocity branch we have

$$\gamma_{\mathbf{k}}^{\rm nl+} = \gamma_{\mathbf{k}}^{\rm l} + \delta\gamma_{\mathbf{k}}^{\rm TSW} + \delta\gamma_{\mathbf{k}}^{\rm POL+} \tag{3.5}$$

Here $\gamma_{\mathbf{k}}^{\rm l}$ is the Landau damping (plus possible collisional damping) arising from the linear dielectric in Eq. (2.3); $\delta\gamma_{\mathbf{k}}^{\rm TSW}$ is the contribution from the direct third-order susceptibility $\varepsilon^{(3)}$ and $\delta\gamma_{\mathbf{k}}^{\rm POL}$ is the contribution from the "polarization cloud" term involving $\varepsilon^{(2)}\varepsilon^{(2)}/\varepsilon_{k-k'}^{\rm nl}$.

The contribution $\delta\gamma_{\mathbf{k}}^{\rm TSW}$ was first discussed in relation to the stability of Langmuir waves by Tsytovich, Stenflo, and Willhelmsonn[5] and can be reduced to the expression

$$\frac{\delta\gamma_{\mathbf{k}}^{\rm TSW}}{\omega_{\mathbf{k}}^{\rm l}} = 3\,\frac{e^2}{m^2\omega^2} \int d^3k' \int d\omega'\, (\mathbf{k}\cdot\mathbf{k}')(k')^2 \langle |\phi|^2_{\mathbf{k}'\omega'} \rangle \operatorname{Im} \chi_e(\mathbf{k}', \omega') \tag{3.6}$$

where $\chi_e(k', \omega')$ is the familiar linear electron susceptibility

$$\chi_e(\mathbf{k}, \omega) = \frac{4\pi e^2}{mk^2} \int d\mathbf{v}\, (\omega - \mathbf{k}\cdot\mathbf{v} + i\delta)^{-1}\, \mathbf{k}\cdot\partial_v \langle f^e(\mathbf{v})\rangle \tag{3.6a}$$

The ordering

$$\omega \gg \omega' \quad \text{and} \quad \omega \gg |\mathbf{k}'\cdot\mathbf{v}|, \quad |\mathbf{k}\cdot\mathbf{v}|, \quad |(\mathbf{k}-\mathbf{k}')\cdot v| \tag{3.7}$$

was assumed in deriving this expression. Strong turbulence corrections to this formula will be discussed below. It is easy to show that $\delta\gamma^{\rm TSW}_{-\mathbf{k}} = \delta\gamma^{\rm TSW}_{\mathbf{k}}$ as always required. More importantly it can be shown that for the negative phase velocity branch the sign of the $\delta\gamma^{\rm TSW}$ term is reversed:

$$\gamma^{\rm nl-}_{\mathbf{k}} = \gamma^{\rm l}_{\mathbf{k}} - \delta\gamma^{\rm TSW}_{\mathbf{k}} + \delta\gamma^{\rm POL-}_{\mathbf{k}} \tag{3.8}$$

At this stage we see that one phase velocity branch is destabilized by the $\delta\gamma^{\rm TSW}$ contribution while the other is stabilized. Equation (3.6) can be made more specific by introducing the following ansatz for the frequency dependence of the ion acoustic potential fluctuation spectrum:

$$\pi\langle|\phi|^2_{\mathbf{k},\omega}\rangle = I^{s+}_{\mathbf{k}} \frac{|\tilde{v}^{s+}_{\mathbf{k}}|}{(\omega - \tilde{\omega}^{s+}_{\mathbf{k}})^2 + (\tilde{v}^{s+}_{\mathbf{k}})^2} + I^{s-}_{\mathbf{k}} \frac{|\tilde{v}^{s-}_{\mathbf{k}}|}{(\omega - \tilde{\omega}^{s-}_{\mathbf{k}})^2 + (\tilde{v}^{s-}_{\mathbf{k}})^2} \tag{3.9}$$

where $I^{s\pm}_{\mathbf{k}}$ are the $\mathbf{k}$ space spectral densities of the electrostatic potential fluctuations for positive and negative phase velocity *ion sound waves* in the turbulent system. Here $\omega^{s\pm}_{\mathbf{k}} = \pm|\mathbf{k}|\, c_s$ are the sound wave frequencies of the two branches and $\tilde{v}^{s\pm}_{\mathbf{k}}$ the corresponding *growth* rates corresponding to the zeroes of $\varepsilon^{\rm nl}(\mathbf{k}, \omega^{s\pm}_{\mathbf{k}} + iv^{s\pm}_{\mathbf{k}}) = 0$. Note we always have the convention $\omega^{s\pm}_{-\mathbf{k}} = -\omega^{s\pm}_{\mathbf{k}}$. With this ansatz and assuming $|\omega^s_{\mathbf{k}}| \gg |v^{s\pm}_{\mathbf{k}}|$ and $|v^{s\pm}_{\mathbf{k}}| \ll kv_e$ we can carry out the ω' integration in Eq. (3.6) to obtain

$$\frac{\delta\gamma^{\rm TSW}_k}{\omega^{\rm l}_{\mathbf{k}}} \simeq -3\frac{e^2}{T^2} \int d^3k' \frac{(\mathbf{k}\cdot\mathbf{k}')}{k_D^2} \left(I^{s+}_{\mathbf{k}'} \frac{v^{e+}_{\mathbf{k}'}}{\omega^{s+}_{\mathbf{k}'}} + I^{s-}_{\mathbf{k}'} \frac{v^{e-}_{\mathbf{k}'}}{\omega^{s-}_{\mathbf{k}'}} \right) \tag{3.10}$$

where $v^{e\pm}_{\mathbf{k}'}$ is the *electron contribution* to the ion acoustic growth rate

$$\frac{\tilde{v}^{e\pm}_k}{\omega^{s\pm}_{\mathbf{k}}} = \frac{1}{2}\frac{k^2}{k_0^2} \operatorname{Im} \chi_e(\mathbf{k}, \omega^{s\pm}_{\mathbf{k}}) \tag{3.11}$$

It is easy to see from Eq. (3.7) that if $\langle f_e(\mathbf{v})\rangle = \langle f_e(-\mathbf{v})\rangle$ that $\delta\gamma^{\rm TSW}$ vanishes when $I^{s+} = I^{s-}$ since $v^{e+}_{\mathbf{k}'} = v^{e-}_{\mathbf{k}'}$. More significantly for a drifted Maxwellian distribution we have for $T_e \gg T_i$

$$v^{e\pm}_{\mathbf{k}}/\omega^{s\pm}_{\mathbf{k}} = (\pi/8)^{1/2}(\mathbf{k}\cdot\mathbf{u} - \omega^{s\pm}_{\mathbf{k}})/(kv_e) \tag{3.12}$$

and for $u \gg c_s$ we see that $\nu_{\mathbf{k}}^{e-}/\omega_{\mathbf{k}}^{s-} = \nu_{\mathbf{k}}^{e+}/\omega_{\mathbf{k}}^{s+}$ and both terms in Eq. (3.10) are destabilizing for positive Langmuir phase velocities. However, in the drift case only positive phase velocity ion acoustic waves will be excited and we will have $I_{\mathbf{k}}^{s+} \gg I_{\mathbf{k}}^{s-}$, which is also the most unstable case for the symmetric velocity distribution.

For the remainder of this article we will assume maximal anisotropy and take $I_{\mathbf{k}}^{s-} = 0$. In this case when the integral of the first term in Eq. (3.10) is positive then Langmuir waves with positive phase velocities will be destabilized by $\delta\gamma^{\text{TSW}}$.

It can be shown that Eq. (3.9) is a valid representation for the spectrum both in the growing stage of the turbulence and in the quasistationary state where $\partial_t I_{\mathbf{k}}^{s\pm} \ll 2\,|\tilde{\nu}_{\mathbf{k}}^{s\pm}|\, I_{\mathbf{k}}^{s\pm}$; in the latter case $\nu_{\mathbf{k}}^{s} \leq 0$, corresponding to stable ion acoustic excitations. For the Markovianized theory which we will use below to be valid in the growing state it is necessary[9] that $\tilde{\nu}_{\mathbf{k}'}^{s}\tau_c \ll 1$, where τ_c is a spectral correlation time $\tau_c^{-1} \sim 3\,|\mathbf{k}'-\mathbf{k}|\,|\Delta k'|\,\lambda_D^2\omega_p$, where $\lambda_D = k_D^{-1}$ is the electron Debye length and $|\Delta k'|$ is the width in wavevector space of the ion acoustic spectrum. The theories used here are not generally valid for coherent ion acoustic waves with $\Delta k' = 0$.

An estimate of the magnitude of $\delta\gamma^{\text{TSW}}$ can be obtained from the following formula:

$$\frac{\delta\gamma_k^{\text{TSW}}}{\omega_p} \simeq -3\langle k'\lambda_D\rangle\, k\lambda_D \langle|\cos\theta|\rangle \frac{W^s}{nT}\frac{1}{1+\langle(k'\lambda_D)^2\rangle}\left\langle\frac{\nu^e}{\omega^s}\right\rangle \tag{3.13}$$

where we have used the definition of the total energy in sound waves

$$\frac{W^s}{nT_e} = \frac{e^2}{T_e^2}\int d^3k'\, I_{k'}^s[1+(k'\lambda_D)^2] \tag{3.14}$$

and the angular brackets indicate a suitable average over the spectrum with $\cos\theta \equiv \hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'$. [For most of the estimates made in this article we will ignore the $(k'\lambda_D)^2$ term in the integrand of Eq. (3.14) since $k'\lambda_D \lesssim 1$.]

It is useful to compare the contribution $\delta\gamma_{\mathbf{k}}^{\text{TSW}}$ with the linear Landau damping $\gamma_{\mathbf{k}}^{1} \neq \omega_p = (\pi/8)^{1/2}(k_D/k)^3 \exp[-(k_D^2/2k^2 + 3/2)]$. The combination $\gamma_{\mathbf{k}}^{1} + \delta\gamma_{\mathbf{k}}^{\text{TSW}}$ for positive phase velocity Langmuir waves reaches its maximum negative (unstable) value at $k = k_f$, where $0.2 < k_f\lambda_D < 0.3$ for $W^s/nT < 10^{-2}$, say. For larger $k\lambda_D$ Landau damping dominates and stabilizes the system. This is shown schematically in Fig. 1.

The role of the polarization contributions $\delta\gamma_{\mathbf{k}}^{\text{POL}\pm}$ has been of some dispute in the literature.[3] In a previous paper[9] we showed that for the

[3] M. Nambu[20] claimed that the polarization terms produced a strong destabilizing contribution which was subsequently shown to be zero in Ref. 9 and by A. Hirose.[21] In Ref. 9 a

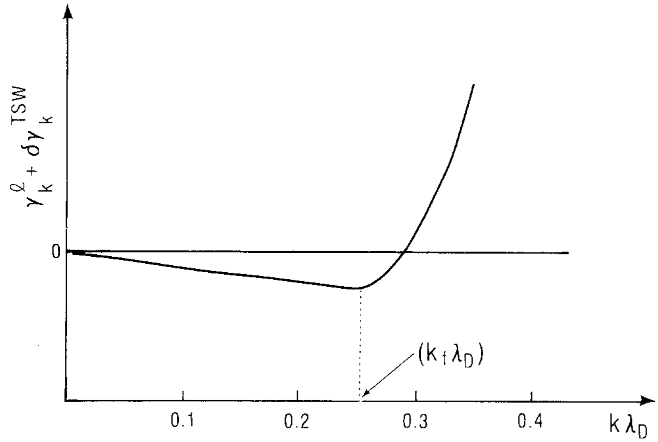

Fig. 1. Linear Landau damping, γ^{l}_{k}, plus the "turbulent bremsstrahlung" rate, $\delta\gamma^{TSW}_{k}$, as a function of $k\lambda_D$ showing the region of *potential* Langmuir wave instability for $k\lambda_D \lesssim 0.3$ and the region of strong Landau damping for $k\lambda_D \gtrsim 0.3$.

growing phase of ion acoustic turbulence, where the weak turbulence approximations are most likely to be valid, resonant and nonresonant decay processes—where a Langmuir wave decays into another Langmuir wave by emitting or absorbing an acoustic wave—are stabilizing for Langmuir waves and always dominate the TSW destabilizing process. The polarization contribution can be calculated as in Ref. 9 when k, $k' \ll k_D$ to give

$$\frac{\delta\gamma^{\mathrm{POL}\pm}_{k}}{\omega_p} = \frac{1}{2}\frac{e^2}{T^2}\int d^3k' \int d\omega' \, (\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2 \, \mathrm{Im}\,\frac{1}{\varepsilon^{\mathrm{nl}}(\mathbf{k}'', \omega'')} \, \langle|\phi|^2_{\mathbf{k}'\omega'}\rangle \quad (3.15)$$

where $\mathbf{k}'' \equiv \mathbf{k} - \mathbf{k}'$ and $\omega'' \equiv \omega - \omega' = \omega^{l\pm}_{\mathbf{k}} - i\gamma^{l\pm}_{\mathbf{k}} - \omega'$. In this article a single prime will always denote ion acoustic wave vectors and double primes Langmuir wave vectors. The ω' integration can be carried out using the ansatz (3.9) with $I^{s-}_{\mathbf{k}'} = 0$ to give

$$\delta\gamma^{\mathrm{POL},\sigma}_{\mathbf{k}}/\omega^{l}_{\mathbf{k}} = \sum_{\sigma''}\int d^3k' \, M^{\sigma\sigma''}_{\mathbf{k},\mathbf{k}''}$$

subdominant destabilizing contribution arising from the polarization terms was calculated and called $\delta\gamma^{x}_{\mathbf{k}}$; this was dominated by the nonresonant stabilizing contribution $\delta\gamma^{\mathrm{NR}}_{\mathbf{k}}$ arising from decay interactions. We have subsequently found that $\delta\gamma^{x}_{\mathbf{k}} = 0$, which makes the Langmuir waves even more stable than we originally thought, the only remaining destabilizing term being $\delta\gamma^{\mathrm{TSW}}_{\mathbf{k}}$ arising from the direct $\varepsilon^{(3)}$ term. We thank Dr. Nambu for pointing out our error; the correct result $\delta\gamma^{x}_{\mathbf{k}} = 0$ *strengthens* our conclusion that Langmuir waves are stabilized by nonresonant decay process. J. Kuipers[22] considered polarization effects but did not consider the competitive role of the nonresonant decay processes.

where

$$M^{\sigma,\sigma''}_{\mathbf{k},\mathbf{k}''} \equiv \frac{1}{4}\frac{e^2}{T^2}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2 I^{s+}_{\mathbf{k}'} \frac{\omega_p \Gamma^{\sigma\sigma''}}{(\Delta\omega^{\sigma\sigma''})^2+(\Gamma^{\sigma\sigma''})^2} \tag{3.16}$$

and where σ, $\sigma''=\pm$ signify the sign of the phase velocity for Langmuir waves with $\mathbf{k}$, $\mathbf{k}''$, respectively,

$$\begin{aligned} \Delta\omega^{\sigma\sigma''} &= \omega^{\mathrm{nl}\sigma}_{\mathbf{k}} - \omega^{\mathrm{nl}\sigma''}_{\mathbf{k}''} - \omega^{s}_{\mathbf{k}'} \\ \Gamma^{\sigma\sigma''} &= \gamma^{\mathrm{nl}\sigma}_{\mathbf{k}} + \gamma^{\mathrm{nl}\sigma''}_{\mathbf{k}''} + \tilde{\nu}^{s}_{\mathbf{k}'} \end{aligned} \tag{3.17}$$

Here we have explicitly included the contributions to the ω' integration from both the positive and negative phase velocity zeroes of $\varepsilon^{\mathrm{nl}}(\mathbf{k}'', \omega'')$.

Because of the occurrence of $\gamma^{\mathrm{nl},\sigma}$ in the DIA expression for $\delta\gamma^{\mathrm{POL}}_{\mathbf{k}}$ the relation [see Eq. (3.5)] $\gamma^{\mathrm{nl}} = \gamma^{\mathrm{l}} + \delta\gamma^{\mathrm{TSW}} + \delta\gamma^{\mathrm{POL}}\{\gamma^{\mathrm{nl}}\}$ is actually an integral equation for γ^{nl}.

The occurrence of $\gamma^{\mathrm{nl},\sigma''}_{k''}$ in the expression (3.17) for $\Gamma^{\sigma\sigma''}$ requires special attention for broad ion acoustic spectra, such as in the Horton–Choi theory, where $k'\lambda_D$ and therefore $|\mathbf{k}''\lambda_D| = |\mathbf{k}-\mathbf{k}'|\,\lambda_D$ can be large enough to be in the strong Landau damping regime. For $|k''|\,\lambda_D > k_c\lambda_D$, where $0.4 \gtrsim k_c\lambda_D \gtrsim 0.3$ the linear Landau damping $\gamma^{\mathrm{l}}_{\mathbf{k}''}$ becomes so large as to overwhelm the other terms in (3.17); i.e., for $|\mathbf{k}''| > k_c$ we have $\gamma^{\mathrm{nl}}_{\mathbf{k}''} \simeq \gamma^{\mathrm{l}}_{\mathbf{k}''} \gg \gamma^{\mathrm{nl}}_{\mathbf{k}}$ or $\tilde{\nu}^{s}_{k'}$. We also find that when $k'' > k_c$ then $\gamma^{\mathrm{l}}_{k''} \gg \Delta\omega^{\sigma\sigma''}$ (explicit expressions for the mismatch are given below); thus the resonance function in Eq. (3.16), $\omega_p\Gamma^{\sigma\sigma''}[(\Delta\omega^{\sigma\sigma''})^2 + (\Gamma^{\sigma\sigma''})^2]^{-1} \simeq \omega_p/\gamma^{\mathrm{l}}_{k''}$, which is relatively small compared to the value when $|k''| < k_c$. We will take this resonance cutoff effect into account by restricting the region of integration over $\mathbf{k}'$ to R:

$$R\colon |\mathbf{k}-\mathbf{k}'| < k_c \tag{3.18}$$

If the ordering in Eq. (3.7) is strictly observed then this restriction is unnecessary. But for spectra arising in the drift-generated ion acoustic turbulence such as Horton–Choi we must violate this ordering. In addition to the effect on the polarization term considered above there will also be a contribution from the direct ($\varepsilon^{(3)}$) term in Eq. (2.3) which arises from the $\omega'' = \mathbf{k}''\cdot\mathbf{v}$ resonance. This contribution corresponds to a kinetic effect which can be described as induced conversion of Langmuir waves into ion acoustic waves by scattering from electrons. For a mean distribution $\langle f^e(\mathbf{v})\rangle$ which has negative slope for $v > v_e$ (e.g., a Maxwellian) this contribution will be stabilizing. We will not give the explicit formulas for this contribution here but rather concentrate on other stabilizing effects which arise for $k'\lambda_D \ll 1$. Thus in Eq. (3.6) or (3.10) we will take the $\mathbf{k}'$ integration over the restricted region R. It is important to keep in mind, however, that for a broad spectrum where $k'_{\max}\lambda_D \sim O(1)$ there exists this additional stabilizing effect.

3.1.1. Resonant Decay Contributions. The resonant decay contributions will be important when ion sound wave vectors $\mathbf{k}'$ which satisfy the condition $\Delta\omega^{\sigma\sigma''}=0$ fall within the support of the ion acoustic spectrum. Since $|\omega^s_{\mathbf{k}'}| \ll \omega_p$ the resonance condition requires $\omega^{1\sigma}_{\mathbf{k}} \simeq \omega^{1\sigma''}_{\mathbf{k}''}$, implying $|\mathbf{k}| \simeq |\mathbf{k}''|$, and because of the factor $(\hat{k}\cdot\hat{k}'')^2$ either forward or backward scattering is favored. Forward scattering implies $|k'| \simeq 0$; because the phase volume for $k' \simeq 0$ is so small and because the spectrum $I^s_{k'}$ cuts off for small k' forward scattering is suppressed for $k > k_* = \frac{2}{3}(m_e/m_i)\, k_{De}$. In backscattering $\mathbf{k}'' = \sim -\mathbf{k}$, so to ensure $\omega^{1\sigma}_k \cong \omega^{1\sigma''}_{-\mathbf{k}}$ we require that $\sigma'' = -\sigma$, corresponding to a backscattered wave with negative phase velocity. *Thus resonant decay couples a positive phase velocity Langmuir wave with a negative phase velocity Langmuir wave.* This fact has particularly important consequences in the wave-kinetic problem considered below.

3.1.1a. Weak Turbulence Formula for Resonant Decay. The familar weak turbulence expression for the resonant decay rate is obtained when the spectrum $I^{s+}_{k'}$ is wide enough in $\mathbf{k}'$ space that the resonance function in the integrand of Eq. (3.16) can be replaced by a delta function. This condition can be expressed as

$$\Gamma^{+-}\tau_c \ll 1 \tag{3.19a}$$

where τ_c is the spectral correlation time

$$\tau_c^{-1} \simeq \frac{\partial \Delta\omega^{+-}}{\partial k'}\, |\Delta k'| \simeq \frac{3}{2}\, |2k\cdot\hat{k}' - 2k' - k_*|\; |\Delta k'|\; \lambda_D^2 \omega_p \tag{3.19b}$$

where $|\Delta k'|$ is the width of the spectrum $I^{s+}_{k'}$. Then we can approximate $\delta\gamma^{\mathrm{POL}}_{\mathbf{k}}$ as

$$\frac{\delta\gamma^{\mathrm{POL}+}_{\mathbf{k}}}{\omega^1_k} \simeq \frac{\delta\gamma^{\mathrm{RES}}_{\mathbf{k}}}{\omega^1_k} = \frac{1}{8}\int_R d^3k'\, (\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2\, \frac{e^2}{T^2}\, I^{s+}_{\mathbf{k}'}\, \omega_p \pi \delta(\Delta\omega_{+-}) \tag{3.20}$$

where

$$\Delta\omega_{+-} = \omega^{1+}_k - \omega^{1-}_{k''} - \omega^s_{k'} \tag{3.20a}$$

The condition $\Delta\omega_{+-} = 0$ reduces to

$$\frac{2}{3}\frac{\Delta\omega_{+-}}{\omega_p} = (2\mathbf{k}\cdot\mathbf{k}' - (k')^2 - k'k_*)\, \lambda_D^2 = 0 \tag{3.21}$$

where $k_* \equiv \frac{2}{3}(m_e/m_i)^{1/2} k_{De}$. Equation (3.20) is valid provided the resonant values of $\mathbf{k}'$ fall within the spectrum

$$k'_{\min} < 2\mathbf{k} \cdot \hat{\mathbf{k}}' - k_* < \Delta k' = \min(k'_c, k'_{\max}) \tag{3.22}$$

where $k'_{\min}$, $k'_{\max}$ are, respectively, the maximum and minimum wave numbers of the spectrum $I^{s+}_{\mathbf{k}'}$.

Because of the delta function in the integrand of Eq. (3.20), $\delta\gamma_k^{\mathrm{RES}}$ is quite sensitive to the shape of the spectrum $I^s_{\mathbf{k}'}$. We can evaluate Eq. (3.20) approximately in several cases to find

$$\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{RES}}}{\omega_k^1} \simeq \frac{\pi}{6} \frac{W^s}{nT} \frac{kk_D^2}{(\Delta k')^3} Z(\Delta k'/2k) \tag{3.23}$$

Here $\Delta k'$ is the smaller of the maximum wave number in the ion acoustic spectrum or the cutoff wave number k'_c, and we assume $k'_{\max}$, $k'_c \gg k'_{\min}$, the lowest wave number in the spectrum. When $k'_c < k'_{\max}$ then W^s is replaced by $W^s_R = (e^2/T^2) \int_R d^3k'\, I^s_{k'}$.

(i) For a one-dimensional spectrum $I^s_{\mathbf{k}'} = I^s_{k'_x} \delta(k'_y)\, \delta(k'_z)$ if $I^s_{k'_x}$ is a smooth function of k'_x we find $Z(y) = 2y^2$ if $y = \Delta k'/2k > 1$, and $Z(y) = 0$ otherwise.

(ii) If we assume that the angular dependence can be factored,[3] $I^s_{\mathbf{k}'} = I^s_{|k'|} P(\cos\theta)$, where $\cos\theta = \hat{\mathbf{k}}' \cdot \hat{\mathbf{u}}$, then for a broad angular spectrum where $(\cos\theta)_{\min} \simeq 0$ we find $Z(y) = 1$ for $y > 1$ and $Z(y) = (\frac{1}{2})[1 - (1 - 2y^2)^3]$ for $y < 1$.

(iii) For a narrow angular spectrum of the factorable type considered in (ii) where $(\cos\theta)_{\min} \simeq 1$ we find $Z(y) = \frac{2}{3}$ for $y > 1$ and $Z(y) = 0$ for $y < 1$.

In all cases $Z \gtrsim O(1)$ for $2k < \Delta k'$ and is smaller or zero for $2k > \Delta k'$. With these estimates and the assumption that $\gamma_{\mathbf{k}}^{\mathrm{nl}} \sim \delta\gamma_{\mathbf{k}}^{\mathrm{RES}} \gg |\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}|$ the condition for the weak turbulence formula leads to $\Gamma\tau_c \leqslant 2\delta\gamma_{\mathbf{k}}^{\mathrm{RES}}\tau_c \ll 1$ or

$$\frac{W^s}{nT_e} \ll \frac{9}{\pi} (\Delta k' \lambda_p)^4 Z^{-1} \tag{3.24}$$

for $k > k_*/2$.

For the modified Kadomtsev spectrum calculated by Horton and Choi[3] we have $k'_{\max}\lambda_D \lesssim 1$, $\langle k'\lambda_D \rangle \sim 0.3$–$0.5$ and the lower cutoff determined by small angle scattering of electrons off ion acoustic waves in $k'_{\min}\lambda_D \sim (W^s/nT)$. The condition (3.22) is well satisfied by k near the fastest-growing mode k_f for which $\gamma_k^1 + \delta\gamma_k^{\mathrm{TSW}}$ has its most negative value. Using Eqs. (3.13) and (3.23) we find the ratio

$$\left|\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}}{\delta\gamma_{\mathbf{k}}^{\mathrm{POL}}}\right| \simeq \left|\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}}{\delta\gamma_{\mathbf{k}}^{\mathrm{RES}}}\right| \simeq \frac{18}{\pi}\frac{(\Delta k'\lambda_D)^3\langle k'\lambda_D\rangle}{Z}\langle v^e/\omega^s\rangle\langle|\cos\theta|\rangle \tag{3.25}$$

which is generally much less than unity. Until further notice angular brackets $\langle\cdots\rangle$ will denote some appropriate average of the indicated quantity over the spectrum $I_{k'}^{s+}$. Thus the stabilizing rate for resonant decay, when allowed, will overwhelmm the destabilizing rate for the TSW process.

For small k where

$$2k < k'_{\min} + k_* \tag{3.26}$$

resonant decay is not allowed, while $\delta\gamma_k^{\mathrm{TSW}}$ is small but nonzero. In the case of a spectrum for which $2k_f > k'_{\max}$ (which is not the Horton–Choi case) again resonant decay is not allowed or is weak. For these cases we need to consider the contribution from nonresonant decay processes, which we do in Section 3.1.2 below.

3.1.1b. Strong Turbulence Limit for $\delta\gamma_k^{\mathrm{POL}}$. For a narrow ion acoustic spectrum where $|\Delta k'|\,\lambda_D \ll 1$ the weak turbulence condition (3.24) may be violated. The strong turbulence regime results when the resonance width Γ^{+-} exceeds the maximum mismatch $\Delta\omega^{+-}$. Then if we assume that $\Gamma^{+-} \simeq 2\gamma^{\mathrm{nl}+} \gg |\Delta\omega^{+-}|$, where $\gamma^{\mathrm{nl}+}$ is weakly dependent on k, and that $\gamma^{\mathrm{nl}+} \sim \delta\gamma^{\mathrm{POL}+}$, we find the simple equation

$$\frac{\gamma^{\mathrm{nl}+}}{\omega^{\mathrm{l}}} \sim \frac{\delta\gamma^{\mathrm{POL}+}}{\omega^{\mathrm{l}}} \sim \frac{1}{4}\frac{e^2}{T^2}\int_R d^3k'\,\langle\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}''\rangle^2 I_{\mathbf{k}'}^{s+}\,\frac{\omega_p}{\gamma^{\mathrm{nl}}} \tag{3.27}$$

The contributions involving $\Delta\omega^{++}$ for sgn $k'' < 0$ and $\Delta\omega^{+-}$ for sgn $k'' > 0$ will be smaller by a factor $\gamma^{\mathrm{nl}}/\omega_p$ and can be neglected compared to the contributions from ω^{+-} for sgn $k'' < 0$ and from ω^{++} for sgn $k'' > 0$ which combine to give Eq. (3.27).

This equation has the simple solution

$$\frac{\gamma^{\mathrm{nl}}}{\omega^{\mathrm{l}}} \sim \frac{1}{2}\left[\frac{e^2}{2T^2}\int d^3k'\,(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2 I_{\mathbf{k}'}^{s+}\right]^{1/2} \tag{3.28}$$

It can be shown by continuity arguments that only the positive square root is permissible.

The k' integration is easily estimated to give

$$\frac{\gamma^{\mathrm{nl}}}{\omega_k^{\mathrm{nl}}} \sim \frac{1}{2}\left(\frac{W^s}{2nT_e}\langle\cos^2\theta\rangle\right)^{1/2} \tag{3.29}$$

and in this case the condition $\Gamma^{+-} \sim 2\gamma^{\rm nl} > |\Delta\omega^{\pm}|$ becomes

$$\frac{1}{8}\frac{W^s}{nT}\langle\cos^2\theta\rangle > 9(k'_{\rm max}\lambda_D)^2\langle|\mathbf{k}-\mathbf{k}'|\,\lambda_D\rangle^2 \tag{3.30}$$

which is essentially the converse of (3.24). This may be satisfied for $W^s/nT \ll 1$. This asymptotic strong turbulence result shows that the stabilizing effect of $\delta\gamma_{\mathbf{k}}^{\rm POL}$ is maintained for larger values of W^s/nT. In this asymptotic case

$$\left|\frac{\delta\gamma_{\mathbf{k}}^{\rm TSW}}{\delta\gamma_{\mathbf{k}}^{\rm nl}}\right| \simeq \left|\frac{\delta\gamma_{\mathbf{k}}^{\rm TSW}}{\delta\gamma_{\mathbf{k}}^{\rm POL}}\right| \sim (k\lambda_D)\langle k'\lambda_D\rangle(W^s/nT)^{1/2}\langle v^e/\omega^s\rangle \ll 1 \tag{3.31}$$

as long as $W^s/nT < 1$.

In this strong limit it is easily verified that the shift in the Langmuir frequency is still negligible: We find $(\omega_{\mathbf{k}}^{\rm nl}-\omega_{\mathbf{k}}^{\rm l})/\omega_p \simeq (W^s/nT)^{1/2}\langle\Delta\omega\rangle/\Gamma$, which is is small and weakly dependent on k so differences in Langmuir frequencies such as occur in $\Delta\omega$ are not affected.

3.1.2. Nonresonant Decay Contributions. An intermediate case occurs when $\Delta\omega^{+-}=0$ cannot be satisfied for $\mathbf{k}'$ within the spectrum and yet $|\Delta\omega^{+-}| \gg \Gamma_{+-}$ over the spectrum. In this case we can approximate $\delta\gamma^{{\rm POL},\sigma}$ by

$$\frac{\delta\gamma_k^{{\rm POL},\sigma}}{\omega_k^{\rm l}} \simeq \frac{\delta\gamma_k^{{\rm NR},\sigma}}{\omega_k^{\rm l}} \simeq \frac{1}{4}\frac{e^2}{T^2}\int_R d^3k' \sum_{\sigma''}(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2\, I_{\mathbf{k}'}^{s+}\frac{\omega_p\Gamma^{\sigma\sigma''}}{(\Delta\omega^{\sigma\sigma''})^2} \tag{3.32a}$$

$$\frac{\delta\gamma_k^{{\rm NR},+}}{\omega_k^{\rm l}} \simeq \frac{1}{4}\frac{e^2}{T^2}\int_R d^3k'\,(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2 \times I_{k'}^{s+}\left[\frac{\omega_p\Gamma^{+-}}{(\Delta\omega^{+-})^2}\,\theta(-{\rm sgn}\,k'')+\frac{\omega_p\Gamma^{++}}{(\Delta\omega^{++})^2}\,\Theta({\rm sgn}\,k'')\right] \tag{3.32b}$$

where we have noted that in order for the Langmuir frequencies to have opposite signs in $\Delta\omega^{+-}$ we require ${\rm sgn}\,\mathbf{k}''<0$ and in $\Delta\omega^{++}$ we require ${\rm sgn}\,\mathbf{k}''>0$. The remaining portion of $\mathbf{k}''$ space yields contributions smaller by $O(k'\lambda_D)^4$. For the low-k nonresonant regime of inequality (3.26) discussed above we have $\Delta\omega^{+-}=\Delta\omega^{++}\simeq\frac{3}{2}\omega_p(k'\lambda_D)^2$, while for the high-$k$ nonresonant regime discussed below $2k \gg k'_{\rm max} \gg k_*$, $\Delta\omega^{+-}=\Delta\omega^{++}\simeq 3\omega_p(\mathbf{k}\cdot\mathbf{k}')\,\lambda_D^2$. In both limits $\Delta\omega$ is independent of k_*, which implies that $\delta\gamma_k^{\rm POL+}\simeq\delta\gamma_k^{\rm POL-}$. If we also assume that $\delta\gamma_k^{\rm NR+} \gg |\gamma_k^{\rm l}+\delta\gamma_k^{\rm TSW}|$, then $\gamma_k^{\rm nl+}\simeq \gamma_k^{\rm nl-}\simeq\delta\gamma_k^{\rm NR+}$ and we can write

$$\frac{\gamma_k^{\rm nl+}}{\omega_k^{\rm l}} \simeq \frac{\delta\gamma_k^{\rm NR+}}{\omega_k^{\rm l}} \simeq \frac{1}{4}\int_R d^3k'\,(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2\, I_{\mathbf{k}'}^{s+}\frac{\Gamma\omega_p}{(\Delta\omega)^2} \tag{3.33}$$

where now we have to a good approximation $\Gamma \simeq \gamma_{\mathbf{k}}^{\mathrm{nl}+} + \gamma_{\mathbf{k}''}^{\mathrm{nl}+} + |v_{\mathbf{k}'}^{s}|$. If the weak turbulence condition is satisfied:

$$\frac{1}{4}\frac{e^2}{T^2}\int d^3k'\,(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')^2\, I_{\mathbf{k}'}^{s+}\,\frac{\omega_p^2}{(\Delta\omega)^2} \approx \frac{1}{4}\frac{W^s}{nT}\,\langle\cos^2\theta\rangle\,\frac{\omega_p^2}{\langle(\Delta\omega)^2\rangle} \ll 1 \qquad (3.34)$$

then the term proportional to $\gamma_{\mathbf{k}}^{\mathrm{nl}+}$ on the right-hand side of Eq. (3.33) can be neglected in comparison to $\gamma_{\mathbf{k}}^{\mathrm{nl}+}$ on the left-hand side. When $\mathbf{k}'' = \mathbf{k} - \mathbf{k}'$ also lies in the nonresonant region and $\gamma_{\mathbf{k}''}^{\mathrm{nl}} \simeq \gamma_{\mathbf{k}}^{\mathrm{nl}}$ we can also neglect $\gamma_{\mathbf{k}''}^{\mathrm{nl}}$ under the same conditions. But when $2k'' > k'_{\min} + k_*$ we should approximate $\gamma_{\mathbf{k}''}^{\mathrm{nl}}$ by $\delta\gamma_{\mathbf{k}''}^{\mathrm{RES}}$ [Eq. (3.23)] which is much larger than $\delta\gamma_{\mathbf{k}''}^{\mathrm{NR}}$.

For evolving ion acoustic turbulence we may have $|v_{\mathbf{k}'}^{s}| > \gamma_{\mathbf{k}''}^{\mathrm{nl}}$ for $\mathbf{k}'$ in the region R. The total ion acoustic growth rate $\tilde{v}_k^s$ is the sum of an electron part and an ion part: $\tilde{v}_k^s = \tilde{v}_k^e + \tilde{v}_k^i$. For evolving turbulence $\tilde{v}_k^e$ is essentially the linear growth rate v_k^e, where $v_k^e/\omega_k^s \sim O(u/v_e)$, while $\tilde{v}_k^i$ is much smaller than v_k^e. Thus if W^s/nT is sufficiently small so that $\gamma_{k''}^{\mathrm{nl}} \simeq \gamma_{k''}^{\mathrm{l}}$ [use Eq. (3.23) for $\delta\gamma_{k''}^{\mathrm{RES}}$] we have the weak turbulence estimate given in Ref. 9:

$$\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{POL}+}}{\omega_k^{\mathrm{l}}} \simeq \frac{\delta\gamma_{\mathbf{k}}^{\mathrm{NR}+}}{\omega_k^{\mathrm{l}}} \simeq \frac{1}{4}\,\frac{\langle\cos^2\theta\rangle}{\langle(\Delta\omega)^2/\omega_p^2\rangle}\,\frac{\langle v_{\mathbf{k}'}^{s} + \gamma_{\mathbf{k}''}^{\mathrm{l}}\rangle}{\omega_p} \qquad (3.35)$$

and

$$\left|\frac{\delta\gamma_k^{\mathrm{TSW}}}{\delta\gamma_k^{\mathrm{POL}}}\right| \sim 12\,\frac{k}{k_*}\,\frac{\langle(\Delta\omega)^2\rangle}{\omega_p^4}\,\frac{\langle v^e\rangle}{\langle v_{\mathbf{k}'}^{s} + \gamma_{\mathbf{k}''}^{\mathrm{l}}\rangle}\,\frac{\langle|\cos\theta|\rangle}{\langle\cos^2\theta\rangle} \qquad (3.36)$$

It is useful to consider the high- and low-k nonresonant regimes separately. In the later $2k \ll k'_{\min} + k_*$ and $(\Delta\omega) \sim \frac{3}{2}\omega_p(k'\lambda_D)^2$. From Eq. (3.36) we find $|\delta\gamma^{\mathrm{TSW}}/\delta\gamma^{\mathrm{POL}}| \ll 1$, i.e., the nonresonant decay scattering-out rate dominates the TSW term so long as $k/k_* < 1$, $\langle k'\lambda_D\rangle \ll 1$ and $\langle v^e\rangle/\langle v_{\mathbf{k}'}^{s} + \gamma_{\mathbf{k}''}^{\mathrm{l}}\rangle$ is not too large. For developing ion acoustic turbulence considered in Ref. 9 we have $\langle v^e\rangle/\langle v^s\rangle \simeq O(1)$ since the nonlinear saturating effect of induced scattering on ions has not yet developed.

For stationary ion acoustic turbulence $\langle v^s\rangle/\langle v^e\rangle$ can become much less than unity because the growth rate v^e is nearly balanced by the ion linear Landau damping $\tilde{v}_{\mathbf{k}}^i$ on the ions as in the Horton–Choi improved version of the Kadomtsev theory.[3] However, note that $\langle v^e\rangle/\langle\gamma_{k''}^{\mathrm{l}}\rangle \simeq (u/v_e)(k'\lambda_D)(m_e/m_i)^{1/2}\langle\gamma_{k''}^{\mathrm{l}}/\omega_p\rangle^{-1}$ is not particularly small even considering the cutoff at $k'' < k_c$, which allows only $\gamma_{k''}^{\mathrm{l}}/\omega_p < 1$. For stronger turbulence levels the renormalization of $\gamma_{k''}^{\mathrm{l}}$ cannot be neglected. For sufficiently strong W^s/nT [as determined by Eq. (3.22)] we have $\gamma_{\mathbf{k}''}^{\mathrm{nl}} \simeq \delta\gamma_{\mathbf{k}''}^{\mathrm{RES}} \gg \gamma_{\mathbf{k}''}^{\mathrm{l}}$ in the expression for Γ in Eq. (3.33). We then have the estimate [still assuming (3.33)]

$$\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{POL}+}}{\omega_k^{\mathrm{l}}} \simeq \frac{\delta\gamma_{\mathbf{k}}^{\mathrm{NR}+}}{\omega_k^{\mathrm{l}}} \simeq \frac{1}{4}\frac{W^s}{nT}\frac{\langle\cos^2\theta\rangle}{\langle(\Delta\omega/\omega_p)^2\rangle}\frac{\langle\delta\gamma_{\mathbf{k}''}^{\mathrm{RES}}\rangle}{\omega_p}$$

$$\sim\frac{\pi}{54}\left(\frac{W^s}{nT}\right)^2\frac{\langle\cos^2\theta\rangle}{\langle(k'\lambda_D)^4\rangle}\frac{\langle k''\rangle k_D^2}{(\Delta k')^3}Z \tag{3.37}$$

where we have used Eq. (3.23) to obtain the second line. The condition $\delta\gamma_{\mathbf{k}''}^{\mathrm{RES}} \gg \tilde{\nu}_{\mathbf{k}'}^s$ is found to be satisfied when the estimates for $\tilde{\nu}^s$ discussed at the end of Section 2 are used. The comparison with $\delta\gamma^{\mathrm{TSW}}$ is

$$\left|\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}}{\delta\gamma_{\mathbf{k}}^{\mathrm{POL}}}\right| \simeq \left|\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}}{\delta\gamma_{\mathbf{k}}^{\mathrm{NR}}}\right| \simeq \frac{162}{\pi}\frac{k}{\langle k''\rangle}\frac{\langle k'\lambda_D^5\rangle\langle\Delta k'\lambda_D\rangle^3}{(W^s/nT)Z}\frac{\langle|\cos\theta|\rangle}{\langle\cos^2\theta\rangle}\left\langle\frac{\nu^e}{\omega^s}\right\rangle \tag{3.38}$$

Here we see that the nonresonant decay process is again stabilizing (i.e., $|\delta\gamma^{\mathrm{TSW}}/\delta\gamma^{\mathrm{POL}}| < 1$) when

$$(k'\lambda_D)^4 \gg \frac{W^s}{nT} \gg \frac{162}{\pi}\frac{k}{\langle k''\rangle}\langle k'\lambda_D\rangle^5\langle\Delta k'\lambda_D\rangle^3\left\langle\frac{\nu^e}{\omega^s}\right\rangle Z^{-1} \tag{3.39}$$

The left-hand side of this inequality is the weak turbulence condition (3.34). These inequalities are readily satisfied since $[k/\langle k''\rangle]\langle\nu^e/\omega_s\rangle \ll 1$. The condition $\Delta\omega \gg \Gamma \sim \delta\gamma_{k''}^{\mathrm{RES}}$ must also be satisfied and yields the condition $(54/\pi)(k'\lambda_D)(k\lambda_D)^2(\Delta k'\lambda_D) \gg W^s/nT$, which is compatible with the condition (3.34) since $k < k'$ in the regime under consideration.

We have shown that the scattering-out rate for resonant or nonresonant decay processes dominates the destabilizing TSW term. Thus from the point of view of the coherent response, Langmuir waves will be stable for a spectrum such as that of Horton and Choi. In Fig. 2 we schematically show the relation of the resonant and nonresonant decay regimes to the ion acoustic spectrum $I_{2k-k_*}^s$, evaluated at $k' = 2k - k_*$, and the growth rate term $\gamma_{\mathbf{k}}^{\mathrm{l}} + \delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}$. An important characteristic of this spectrum is that $k'_{\max}\lambda_D > k_f\lambda_D$, where k_f is the wave number where $\gamma_{\mathbf{k}}^L + \delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}$ is most negative in value.

Another possible spectral shape, in contrast to the Horton–Choi spectrum, is shown in Fig. 3. Here the high k' end of the spectrum is cut off for $k'_{\max}\lambda_D < 2k_f\lambda_D \ll 1$. This might occur if the ion acoustic turbulence is driven by a beam distribution rather than a drifting Maxwellian. In this case the region of nonresonant interaction is at higher values of the range wave numbers for which $\gamma_{\mathbf{k}}^{\mathrm{l}} + \delta\gamma_{\mathbf{k}}^{\mathrm{TSW}} < 0$:

$$k'_{\max} < 2k - k_* \simeq 2k$$

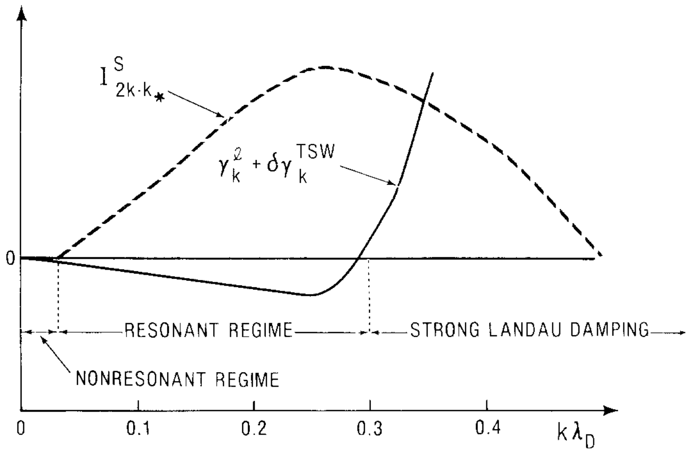

Fig. 2. Schematic representation of the relationship of the ion acoustic spectrum $I^s_{2k-k_*}$ and the potentially destabilizing (i.e., negative) contribution, $\gamma^l_k + \delta\gamma^{TSW}_k$, to the Langmuir damping rate. In the region of $k\lambda_D$ labeled "resonant regime" the positive resonant decay damping $\delta\gamma^{RES}_k$ [Eq. (3.20)] stabilizes Langmuir waves against the relatively small negative damping $\gamma^l_k + \delta\gamma^{TSW}_k$. In the low k region labeled "nonresonant regime" the nonresonant Langmuir wave damping $\delta\gamma^{NR}_k$ [Eq. (3.33)] stabilizes the Langmuir waves, and for $k\lambda_D \gtrsim 0.3$ the linear Landau damping stabilizes the Langmuir waves. This figure applies to the case ion acoustic turbulence driven by a relative electron–ion drift velocity.

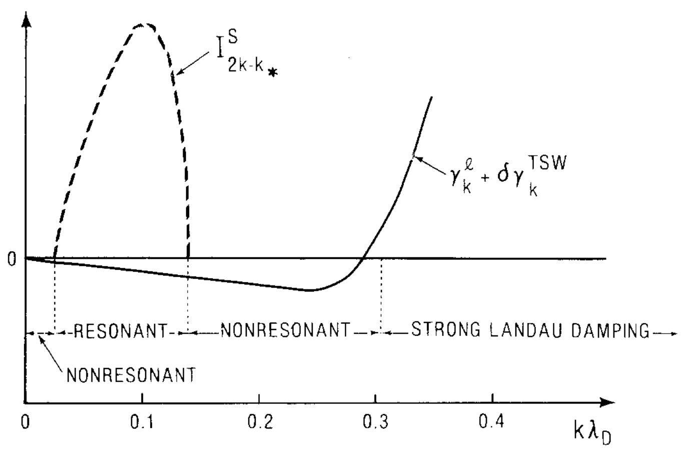

Fig. 3. Schematic representation of the relationship of the ion acoustic spectrum $I^s_{2k-k_*}$ and the potentially destabilizing (i.e., negative) contribution, $\gamma^l_k + \delta\gamma^{TSW}_k$, to the Langmuir damping rate. The comments in the caption to Fig. 2 apply here as well, except in this case there is a nonresonant region at $k_{max}\lambda_D < k\lambda_D \lesssim k_f\lambda_D$, where again the nonresonant Langmuir wave damping $\delta\gamma^{NR}_k$ [Eq. (3.33)] stabilizes the Langmuir waves. This figure applies, for example, to an ion-beam-driven ion acoustic turbulence.

and $(\Delta\omega/\omega_p) \simeq 3(\mathbf{k}\cdot\mathbf{k}')\,\lambda_D^2$. From Eq. (3.36) we find the nonresonant, weak turbulence formula

$$\left|\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}}{\delta\gamma_{\mathbf{k}}^{\mathrm{POL}}}\right| \sim \frac{98}{\pi}\frac{k}{k_*}\langle \mathbf{k}\cdot\mathbf{k}'\lambda_D^2\rangle^2 \frac{\langle v_e\rangle}{\langle \tilde{v}_s\rangle}\frac{\langle|\cos\theta|\rangle}{\langle\cos^2\theta\rangle} \tag{3.40}$$

To evaluate this case we again need to understand the detailed properties of the ion acoustic turbulence. One way to obtain a spectrum such as in Fig. 3 would be with a warm ion beam with velocity $v_B \gtrsim c_s$ and beam width Δv_B. Then $\Delta k' = |k'_{\max} - k'_{\min}| \simeq \langle k'\rangle \Delta v_B/v_B$. In this case in the developing stage the electrons have a (stable) Maxwellian distribution, where $v_k^e \simeq -[(\pi/8)\, m_e/m_i]^{1/2}\,\omega_k^s$. Equation (3.10) for $\delta\gamma^{\mathrm{TSW}}$ then predicts that Langmuir waves with *negative* phase velocities are destabilized by the anisotropy of the ion acoustic spectrum, $I_k^{s+} \gg I_k^{s-}$. Now, however, since $v^s = v^e + v^i$ must be positive in the growing stage of the ion acoustic turbulence, we must have $\tilde{v}_k^s \simeq v_k^i \gg |v_k^e|$; v_k^i is the positive-valued ion Landau growth rate of the beam instability. Thus in the developing stage for this case $|\langle v^e\rangle/\langle v^s\rangle| \ll 1$ and Eq. (3.40) predicts $|\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}/\delta\gamma_{\mathbf{k}}^{\mathrm{POL}}| \ll 1$ so Langmuir waves are again stabilized by the nonresonant decay process. In the long time limit of the ion beam instability we expect that the growth rate $\tilde{v}^i$ will be reduced by quasilinear effects and $\tilde{v}_k^i \to O(|v_k^e|)$ so that $\langle v_k^e\rangle/\langle\tilde{v}^s\rangle \lesssim O(1)$ at long times. This is an example where the TSW effect is potentially destabilizing for a Maxwellian distribution of *electrons* provided $I_{k'}^{s+} \gg I_{k'}^{s-}$. Again for the range of k and k' values typical of Fig. 3, Eq. (3.40) predicts $|\delta\gamma^{\mathrm{TSW}}/\delta\gamma^{\mathrm{POL}}| \ll 1$ and Langmuir waves are stabilized by nonresonant decay interactions. The naive view[23] that one can ignore the competing effects of decay processes in this case is seen to be incorrect, and at least in the case just discussed there is no viable "plasma maser effect."

These applications do not exhaust the possible manifestations of ion acoustic turbulence. The formulas given above are applicable for testing the stability of Langmuir waves in other cases. The main point we wish to make is that even nonresonant decay interactions can be strong enough to stabilize the Langmuir waves against the destabilizing effects of $\delta\gamma_k^{\mathrm{TSW}}$. For any particular manifestation of ion acoustic turbulence, calculations of the competing effects of decay processes must be made.

3.2. Langmuir Wave Kinetics

The stability calculation based on the coherent nonlinear dielectric response $\varepsilon^{\mathrm{nl}}(\mathbf{k}, \omega)$, which has been the approach of all previous work on this subject, is quite incomplete. This approach calculates the mean (i.e., ensemble average) response of the ion acoustic turbulent system to an infinitesimal coherent (i.e., nonzero mean) source or signal with wave vector $\mathbf{k}$ and frequency $\omega \simeq \omega_{\mathbf{k}}^{1}$. As we know from the considerations in Section 2, for example, the response of the fluctuations of the Langmuir fields also involves the incoherent nonlinear source terms—i.e., the terms on the right-hand side of Eq. (2.2). In the case of the decay processes these terms involve decay processes which *scatter-in* Langmuir waves from some wave vector $\mathbf{k}''$ to the wave vector $\mathbf{k}$ whose stability we are investigating. This source term is then *potentially destabilizing*. The coherent response considered above always produced a damping or loss since it takes into account only the inverse process of *scattering out* from wave vector $\mathbf{k}$ to wave vector $\mathbf{k}''$. Thus the condition $\gamma_{\mathbf{k}}^{\mathrm{nl}} > 0$ is only a necessary condition for stability of the fluctuations but is not sufficient because of the scattering-in effect.[24] The kinetic equation for the Langmuir wave intensities can be derived using standard methods from Eq. (2.2). We define in analogy to Eq. (3.9) the Langmuir wave intensities (or actions) for positive and negative phase velocities for $|\omega| \sim \omega_k^{1}$.

$$\langle |\phi|^2_{\mathbf{k},\omega} \rangle = \frac{I_k^{L+}\gamma_k^{\mathrm{nl}+}}{(\omega - \omega_k^{\mathrm{nl}+})^2 + (\gamma_k^{\mathrm{nl}+})^2} + \frac{I_k^{L-}\gamma_k^{\mathrm{nl}-}}{(\omega - \omega_k^{\mathrm{nl}-})^2(\gamma_k^{\mathrm{nl}-})^2} \tag{3.41}$$

where again we take $\omega_k^{\mathrm{nl}\pm}$ as given by Eq. (3.4). The kinetic equations for the slow temporal variational $I_k^{L\pm}$ are given by

$$\partial_t I_k^{L\sigma} + 2\gamma_k^{\mathrm{nl},\sigma} I_k^{L\sigma} = S_k^{L\sigma} \tag{3.42}$$

for $\sigma = +$ or $-$ where the incoherent source term is given in the same approximations which lead to Eq. (3.16) by

$$S_k^{L\sigma} = (2\omega_k^{1}) \int d^3k'\, M_{\mathbf{k},\mathbf{k}''}^{\sigma\sigma''} I_{\mathbf{k}''}^{L\sigma''} \tag{3.43}$$

where $M_{\mathbf{k},\mathbf{k}''}^{\sigma,\sigma''}$ is given in Eq. (3.16), where we showed

$$\frac{\delta\gamma_{\mathbf{k}}^{\mathrm{POL},\sigma}}{\omega_k^{1}} = \int d^3k' \sum_{\sigma''} M_{\mathbf{k},\mathbf{k}''}^{\sigma,\sigma''} \tag{3.44}$$

(We note that there are no incoherent source terms related to the coherent process $\delta\gamma_{\mathbf{k}}^{\mathrm{TSW}}$.) If we write $\gamma^{\mathrm{nl},\sigma}$ according to Eqs. (3.5) and (3.8) it is useful to write separate equations for $+$ or $-$ phase velocities

$$\partial_t I_k^{L+} + 2(\gamma_k^{1} + \delta\gamma_k^{\mathrm{TSW}} + \delta\gamma_k^{\mathrm{POL}+})\, I_k^{L+} = \int d^3k'\, M_{k,k''}^{+\sigma''} I_{k''}^{L\sigma''} \tag{3.45a}$$

$$\partial_t I_k^{L-} + 2(\gamma_k^{1} - \delta\gamma_k^{\mathrm{TSW}} + \delta\gamma_k^{\mathrm{POL}-})\, I_k^{L-} = \int d^3k'\, M_{k,k''}^{-\sigma''} I_{k''}^{L\sigma''} \tag{3.45b}$$

Here we have explicitly accounted for the fact that $\delta\gamma_k^{\mathrm{TSW}+} = -\delta\gamma_k^{\mathrm{TSW}-} = \delta\gamma^{\mathrm{TSW}}$ and a summation over $\sigma'' = +, -$ is implied on the right-hand sides.

The condition of conservation of plasmons in the decay process is expressed by

$$\int d^3k\, 2\delta\gamma_{\mathbf{k}}^{\mathrm{POL},\sigma} I_{\mathbf{k}}^{L\sigma} = \int d^3k \int d^3k'\, M_{\mathbf{k},\mathbf{k}''}^{\sigma,\sigma''} I_{\mathbf{k}''}^{L\sigma''}(2\omega_k^{\mathrm{l}}) \tag{3.46}$$

which follows from Eq. (3.44). Thus we know that the source terms are the same order as $\delta\gamma_k^{\mathrm{POL},\sigma} I_k^{L\sigma}$ so that if $\delta\gamma_k^{\mathrm{POL}}$ is the dominant rate the source terms will also be large. If the source terms have a strong coupling between $+$ and $-$ (or $-$ and $+$) phase velocities then the phase velocity direction *destabilized* by $\delta\gamma^{\mathrm{TSW}}$ will be strongly coupled to the phase velocity direction *stabilized* by $-\delta\gamma^{\mathrm{TSW}}$ and the destabilization effect is greatly reduced. Effectively this coupling reduces $\delta\gamma^{\mathrm{TSW}}$ to $\delta\gamma^{\mathrm{TSW}} \times (\delta\gamma^{\mathrm{TSW}}/\delta\gamma^{\mathrm{POL}}) \ll \delta\gamma^{\mathrm{TSW}}$ when the decay rate dominates.

This effect is most easily calculated for the case of resonant decay when $k'_{\mathrm{min}} < 2k - k_* < k'_{\mathrm{max}}$. If we further restrict considerations to one dimension we find, using Eqs. (3.45a,b), (3.16), (3.20), and (3.23),

$$\int dk'\, M_{k,k''}^{+,-} I_{k''}^{L-} = \delta\gamma_k^{\mathrm{POL}} I_{-(k-k_*)}^{L-} \tag{3.47a}$$

$$\int dk'\, M_{-(k-k_*),k''}^{-,+} I_{k''}^{L+} = \delta\gamma_k^{\mathrm{POL}} I_k^{L+} \tag{3.47b}$$

and $M_{k,k''}^{+,+} = M_{k,k''}^{-,-} = 0$ for *resonant* processes. Here

$$\delta\gamma_k^{\mathrm{POL}} = \frac{\pi}{6}\omega_p \frac{e^2}{T^2} I_{2k-k_*}^{s+} \frac{k_D}{|2k - k_*|} \tag{3.47c}$$

Then using $I_{-(k-k*)}^{L-} = I_{k-k*}^{L-}$ we have the coupled equations

$$\begin{aligned} &\partial_t I_k^{L+} + 2(\gamma_k^{\mathrm{l}} + \delta\gamma_k^{\mathrm{TSW}} + \delta\gamma_k^{\mathrm{POL}})\, I_k^{L+} - 2\delta\gamma_k^{\mathrm{POL}} I_{k-k_*}^{L-} = 0 \\ &\partial_t I_{k-k_*}^{L-} + 2(\gamma_{k-k_*}^{\mathrm{l}} - \delta\gamma_{k-k_*}^{\mathrm{TSW}} + \delta\gamma_{k-k_*}^{\mathrm{POL}})\, I_{k-k_*}^{L-} - 2\gamma_k^{\mathrm{POL}} I_k^{L+} = 0 \end{aligned} \tag{3.48}$$

Because only positive phase velocity ion acoustic waves are excited (by assumption), the resonant decay kinetic equations couple together only modes $\mathbf{k}$ and $\mathbf{k} - \mathbf{k}_*$ with opposite phase velocities in one dimension. The dispersion relation for this set of equations is easily obtained: setting the time dependence of $I_k^{L\pm}$ to go as e^{-2st} we find the most unstable root for $k \gg k_*$ to be

$$s = \gamma_k^1 + \delta\gamma_k^{\mathrm{POL}} - [(\delta\gamma_k^{\mathrm{POL}})^2 + (\delta\gamma^{\mathrm{TSW}})^2]^{1/2}$$
$$\simeq \gamma_k^1 - \frac{1}{2}\frac{(\delta\gamma_k^{\mathrm{TSW}})^2}{\delta\gamma_k^{\mathrm{POL}}} \qquad \text{if} \quad \left|\frac{\delta\gamma^{\mathrm{TSW}}}{\delta\gamma^{\mathrm{POL}}}\right| \ll 1 \tag{3.49}$$

We saw above that the last condition is satisfied in most regimes of interest. The stability condition now is

$$\gamma_k^1 > (\delta\gamma_k^{\mathrm{TSW}})^2/\delta\gamma_k^{\mathrm{POL}} \tag{3.50}$$

so that at this level of approximation very weak Langmuir instability is still possible in principle if the linear dissipation were zero. However, the reduction of the $\delta\gamma_k^{\mathrm{TSW}}$ growth rate by a small factor $|\delta\gamma_k^{\mathrm{TSW}}/\delta\gamma_k^{\mathrm{POL}}| \ll 1$ allows stabilization by small collisional damping or other residual effects. For the fastest growing mode k_f defined in Section 3.1 with respect to $\gamma_k^1 + \delta\gamma_k^{\mathrm{TSW}}$ we have $\delta\gamma_{k_f}^{\mathrm{TSW}}/\gamma_{k_f}^1 \sim O(1)$ so stabilization by the formula (3.50) is assured for this value of $k \sim k_f$.

If the relatively weak nonresonant processes neglected above are considered we find that they are sufficient to stabilize against this greatly reduced "turbulent bremsstrahlung" growth rate. Such nonresonant terms arise, for example, from the contributions from $M_{kk''}^{++}$ or $M_{kk''}^{--}$ in Eqs. (3.44) and (3.45) which couple like sign phase velocities. These nonresonant terms provide access to regions of shorter wavelength where Landau damping provides the ultimate stabilization of the system. As we saw in Section 3.1, the rates for the nonresonant decay processes usually are much larger than $\delta\gamma_k^{\mathrm{TSW}}$. The details of the calculations of the nonresonant contributions and the extension of these arguments to higher dimension will be given elsewhere. The basic physical arguments for the stability of the global wave-kinetic distribution of Langmuir waves hinge on the following facts:

1. The rates for scattering-out and scattering-in due to resonant *or nonresonant* decay processes generally greatly exceed the destabilizing rate $\delta\gamma^{\mathrm{TSW}}$.
2. The scattering-out terms (i.e., the incoherent noise) scatter Langmuir waves into regimes of increased dissipation either by the coupling to opposite phase velocity waves whose stability is increased by $|\delta\gamma^{\mathrm{TSW}}|$ or to higher k'' waves where Landau damping is effective.

3.3. Renormalized Wave–Particle Interactions

The strong turbulence effects considered above involving the renormalized dielectric response $\varepsilon^{\mathrm{nl}}(\mathbf{k}-\mathbf{k}', \omega-\omega')$ in Eq. (2.2) arise from wave–wave interactions. These produced resonance broadening effects which *increased* the rates for decay interactions which stabilized the Langmuir waves.

The "turbulent bremstrahlung" effect giving rise to the destabilizing term $\delta\gamma_k^{\mathrm{TSW}}$ involves a wave–particle resonance, $\omega' = \mathbf{k}' \cdot v$ which occurs in the factor $\mathrm{Im}\,\chi_e(k', \omega')$ in Eqs. (3.6) and (3.7). Choi and Horton and others have shown that the weak turbulence expansion breaks down for ion acoustic turbulence levels where $W^s/nT_e > m_e/m_i$. The primary effect which they considered is the turbulent broadening of the wave–particle resonances. We have studied the renormalization problem using the DIA formulation of plasma turbulence theory—i.e., Theory 3 discussed in Section 2; this theory contains all the other proposed theories.

The analysis starts from the general formula, Eq. (2.7), for the renormalized dielectric response. Since Langmuir wave–particle resonances are weak ($\omega \gg k \cdot v$) we can use the expansion of Eq. (2.10). From this we see that the weak turbulence expression for $\delta\gamma_k^{\mathrm{TSW}}$ arises from the last term in Eq. (2.70), which is proportional to $\mathbf{v}^E_{\mathbf{k},\omega}$. The renormalized TSW term involves only the first direct or nonpolarization term in Eq. (2.13) for $\mathbf{v}^E_{k,\omega}$. Without going into details we state the result that the renormalized limit of this term involves the completely renormalized expression for $v^E_{k,\omega}$ as given for example by the second term in Eq. (B5) in Ref. 15. Using this formula we derive a renormalized expression for $\delta\tilde{\gamma}_k^{\mathrm{TSW}}$:

$$\frac{\delta\tilde{\gamma}_k^{\mathrm{TSW}}}{\omega_k^{\mathrm{l}}} = \frac{4\pi e^3}{m^2}\frac{3}{\omega_p^3}\int d^3k' \int d^3v\, \mathbf{k} \cdot \langle \mathbf{E}_{\mathbf{k}',\omega'} f^e_{-\mathbf{k}',-\omega'}(v) \rangle \tag{3.51}$$

This involves the cross-correlation function between electric field fluctuations $\mathbf{E}_{\mathbf{k}'\omega'} = i\mathbf{k}'\phi_{\mathbf{k}',\omega'}$ and fluctuations in the phase space distribution for electrons, $f^e_{k'\omega'}(v)$. This in turn can be related to diagonal correlations using the general relations[15]

$$\langle f^e_{\mathbf{k}\omega}(v)\, \mathbf{E}_{-k,-\omega} \rangle = -\frac{q_e}{m}\int d\mathbf{v}_1\, g^e_{\mathbf{k},\omega}(\mathbf{v}, \mathbf{v}_1)\, \partial_{\mathbf{v}_1} \tilde{f}^e_{\mathbf{k}\omega}(v_1)\, k^2 \langle |\phi_{k\omega}|^2 \rangle$$
$$+ \frac{k}{ik^2 \varepsilon^{\mathrm{nl}}(\mathbf{k}, \omega)} \sum_j (q_j/\varepsilon_0) \int d\mathbf{v}'\, \langle f^{e,\mathrm{nd}}_{\mathbf{k}\omega}(v)\, f^{j,\mathrm{nd}}_{-\mathbf{k},-\omega}(v') \rangle \tag{3.52}$$

and

$$k^2 \langle |\phi_{\mathbf{k},\omega}|^2 \rangle = \frac{\sum_{jj'} (q_j q_{j'}/\varepsilon_0^2)}{k^2 \varepsilon^{\mathrm{nl}}(\mathbf{k}, \omega)} \int dv \int dv'\, \langle f^{j,\mathrm{nd}}_{\mathbf{k},\omega}(v)\, f^{j',\mathrm{nd}}_{-\mathbf{k},-\omega}(v') \rangle \tag{3.53}$$

These expressions involve the correlation functions for the incoherent or nondiagonal fluctuations as defined in Refs. 15, 14, and 17.

The weak turbulence result of Eq. (3.6) is recovered by making the usual *quasilinear approximation* for $\langle Ef \rangle$, which implies the replacements

i. $\langle f^{j,\mathrm{nd}}_{\mathbf{k}\omega}(v)\, f^{j',\mathrm{nd}}_{-\mathbf{k}-\omega'}(v') \rangle = 0$, i.e., zero incoherent source;
ii. $\bar{f}^{e}_{k\omega}(v) \to \langle f^{e}(v) \rangle$, i.e., no mean field vertex renormalization;
iii. $g_{\mathbf{k}\omega}(\mathbf{v}, \mathbf{v}_1) \to g^{0}_{\mathbf{k}\omega}(v, v_1) = i(\omega - \mathbf{k}\cdot\mathbf{v} + i\delta)^{-1}\, \delta(\mathbf{v} - \mathbf{v}_1)$.

A renormalized theory on the level of that of Horton and Choi is obtained by retaining (i) and (ii) but replacing $g_{\mathbf{k}\omega}(\mathbf{v}, \mathbf{v}_1)$ by the usual resonance broadened propagator $g^{rb}_{\mathbf{k},\omega}(\mathbf{v}, \mathbf{v}_1)$, which is the solution of Eq. (2.9) when the polarization contributions to Eq. (2.11) for v^f are neglected. In this theory Horton and Choi have shown that large-angle turbulent scattering destroys the Landau resonance and cuts off the linear growth rate for $k' < k'_{\min}$, where $k'_{\min}\lambda_{De} \sim (W^s/nT)$. In this approximation we have $\tilde{v}^{e}_{\mathbf{k}'} \lesssim v^{e}_{\mathbf{k}'}$, where $\tilde{v}^{e}_{\mathbf{k}'}$ is a renormalized electron contribution to the ion acoustic growth rate which is determined by the renormalized version of Eq. (3.11):

$$\begin{aligned}\tilde{v}^{e}_{\mathbf{k}}/\omega^{s}_{\mathbf{k}} &\simeq \frac{1}{2}\frac{k^2}{k_D^2}\,\mathrm{Im}\,\tilde{\chi}^{\mathrm{rb}}_{e}(k, \omega_k)\\ &= \frac{1}{2}\frac{k^2}{k_D^2}\frac{4\pi e^2}{mk^2}\int d\mathbf{v}\int d\mathbf{v}'\,\mathrm{Im}\,g^{\mathrm{rb}}_{\mathbf{k},\omega^{s}_{k}}(\mathbf{v}, \mathbf{v}')\mathbf{k}\cdot\partial_v\langle f^{e}(\mathbf{v})\rangle \qquad (3.54)\end{aligned}$$

It is easily shown that in this approximation $\delta\tilde{\gamma}^{\mathrm{TSW}}_{\mathbf{k}}$ is given by Eq. (3.10) with $v^{e\pm}_{\mathbf{k}'}$ simply replaced by the renormalized growth rates $\tilde{v}^{e\pm}_{\mathbf{k}'}$. Since $\tilde{v}^{e}_{\mathbf{k}'} \lesssim v^{e}_{\mathbf{k}'}$ we see that renormalization of the wave–particle resonances *weakens* $\delta\tilde{\gamma}^{\mathrm{TSW}}_{k}$ compared to the weak turbulence prediction. Thus Langmuir waves are *more stable* than we predicted in Sections 3.1 and 3.2 on the basis of the weak turbulence formulas.

We can derive a similar result using the full DIA renormalization. Using Eqs. (3.51)–(3.53) we can show that so long as $\tilde{v}^{s}_{\mathbf{k}}/\omega^{s}_{\mathbf{k}} \ll 1$ $\delta\tilde{\gamma}^{\mathrm{TSW}}$ is given by Eq. (3.10) with $v^{e\pm}_{k'}$ replaced by

$$\tilde{v}^{e\pm}_{k'}/\omega^{s}_{k'} = \frac{1}{2}\frac{k^2}{k_D^2}\,\mathrm{Im}\,\tilde{\chi}_e(k', \omega^{s}_{k'})$$

where now $\tilde{\chi}_e(k'\omega')$ is the complete DIA electron susceptibility

$$\tilde{\chi}_e(k, \omega) = -i\frac{\omega^2_{p_e}}{k^2}\int d\mathbf{v}\int d\mathbf{v}'\, g^{e}_{\mathbf{k},\omega}(\mathbf{v}, \mathbf{v}')\boldsymbol{k}\cdot\partial\bar{f}^{e}_{\mathbf{k},\omega}(v')$$

which is contained in the complete dielectric response of Eq. (2.7). However, in the complete DIA theory we cannot prove as easily that

$\tilde{\nu}^e_{\mathbf{k}'} \lesssim \nu^e_{\mathbf{k}'}$ as in the case of the Choi–Horton theory. The complete $\tilde{\nu}^e_{\mathbf{k}'}$ in the full DIA includes polarization effects not included in the Horton–Choi resonance-broadened theory. At the level of wave–wave resonance broadened weak turbulence theory where the polarization terms are treated perturbatively these polarization terms do provide an additional *damping* due to off-resonant three- (ion acoustic) wave interactions as discussed in Section 2; this effect will again tend to *decrease* the growth rate $\tilde{\nu}^e_{k'}$ from its weak turbulence value.

4. SUMMARY AND COMMENTS

In Section 2 we compared several forms of renormalized plasma turbulence theories with the general framework of Kraichnan's direct interaction theory:

1. The first was the DIA algorithm applied to the nonlinear Poisson equation obtained by an expansion (to cubic order) of the Vlasov phase space distribution in powers of the electrostatic potential fluctuations. This yields a theory first proposed by Tsytovich for the ion acoustic problem which can account for the finite lifetime of ion acoustic excitations in the quasistationary turbulent state. This finite lifetime results from off-resonant three-wave coupling.

2. The second theory was a renormalized theory of the Vlasov equation which assumes that the electrostatic potential fluctuations due to ion acoustic waves obey quasi-Gaussian statistics. This yields the renormalized theory of Horton and Choi plus a mean field "vertex" renormalization not considered by them. This theory (without "vertex renormalization") is the most thoroughly worked out theory of ion acoustic turbulence and agrees at least qualitatively with experiment and numerical simulation.

3. The third theory was the complete DIA applied to the Vlasov equation which contains 1 and 2 plus additional polarization terms which arise from the non-Gaussian statistics of the electrostatic and phase space fluctuations and are consistent with Poisson's equation and the nonlinearity of the Vlasov equation. Both the propagator renormalization and the renormalization of the coupling of fluctuations to the mean field ("vertex renormalization") contain polarization terms whose significance for ion acoustic turbulence remains to be determined.

In Section 3 these theories were applied to the stability of states of ion acoustic turbulence to the excitation of Langmuir waves. The destabilizing "turbulent bremsstrahlung" effect of Tsytovich *et al.* is shown to be dominated in many cases by resonant or nonresonant three-wave decay interactions which stabilize the coherent response for Langmuir excitation.

Renormalization effects due to ion acoustic wave–wave coupling and ion acoustic wave–particle coupling appear to *increase* the stability of Langmuir waves as a result of resonance broadening effects. The global wave kinetics for Langmuir waves was shown to involve the scattering (resonant or nonresonant) of Langmuir waves into regions of increased dissipation, either scattering into oppositely directed phase velocity Langmuir waves which are damped by the TSW effect or into regions of increased (linear) Landau damping.

The results on Langmuir wave stability have an important bearing on the validity of the dissipative Zakharov model, which has been widely used[24,26–28] [4] as a simplified model of the interaction of Langmuir waves and ion acoustic waves. This model consists of the usual Zakharov equations[29] to which linear dissipation is added for both the Langmuir envelope equation and the ion acoustic equation. The turbulent bremsstrahlung process of TSW is not included in the Zakharov model, while the decay processes—resonant and nonresonant—are included.[27] The nonresonant processes particularly depend on the additional dissipations terms in this model, at least in the weak turbulence limit. The fact that we find the decay processes to dominate the turbulent bremsstrahlung effect gives support to the usefulness and validity of the dissipative Zakharov model and is consistent with the usual arguments[29] for the validity of the dissipationless Zakharov equations.

The conventional wisdom[30] has been that short-scale ion acoustic (i.e., density) fluctuations provide an effective damping for long-wave length Langmuir waves. The nonresonant decay results in Section 3.1.2 for $2k < k'_{\min}$ pertain precisely to this regime. In addition we have demonstrated the stabilization of Langmuir waves by ion acoustic turbulence in the resonant regime $k'_{\min} < 2k < k'_{\max}$ and in the regime $k'_{\max} < 2k < 2k_f$ at least in certain cases. The consideration of strong turbulence effects is also new as far as we know.

Recent one-dimensional kinetic simulations[31] and numerical solutions of the dissipative Zakharov model[28] indicate that the picture developed here may be incomplete when the scales of the Langmuir waves and the ion acoustic waves are about equal, i.e., $k \sim k'$. These studies showed[28] that the local Langmuir wave-packets can be nucleated in preexisting ion density wells when a source of Langmuir energy (e.g., a pump field or beam instability) is present at longer wavelengths. This nucleation is just the coherent, near-resonant driving by the source of the local bound Langmuir state in the density cavity; the scales of this bound wave function and the cavity are nearly equal. The effective dissipation of

[4] For a recent review of this subject see Ref. 25.

the long-wavelength driving source by a random collection of short-scale density cavities may be adequately described by the theory discussed in this article. However, it is not clear that the incoherent turbulence theory can take into account the phase coherence observed in the local nucleation effect.

The problem with the turbulence theory for $k \sim k'$ can be seen in the fact that the correlation time in Eq. (3.19) becomes large when $k = k'$. For $k \ll k'$, for example, the correlation time is relatively short and a Langmuir wave experiences an essentially random density field. For $k \sim k'$ the correlation time is much longer, and the Langmuir wave (at k) experiences a more coherent density field. From the Zakharov model for a prescribed, nearly Gaussianly distributed ensemble of density fluctuations one can show[27] quite rigorously that the weak turbulence results refered to in the text, i.e., those results which do not involve any renormalization, are valid provided $|\Delta k'|^2 \lambda_D^2 \omega_p \tau_c \ll 1$ and $[\langle \delta n^2 \rangle^{1/2}/n]\, \omega_p \tau_c \ll 1$ which can also be written as $(W^s/nT) \ll 9\, |\Delta k' \lambda_D|^2 \langle k - k' - k_* \rangle^2$ (in one dimension). [This is seen to be consistent with Eqs. (3.24) and (3.34) for example.] For $k \sim k'$ these conditions are very restrictive and use of the renormalized theory becomes imperative. There is no proof that the renormalized theory will be accurate, however, and much more research is necessary to establish the limits of validity of such theories.

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